Trigonometric Substitutions for the Evaluation of Definite Integrals

The substitution $t = \tan\left(\frac{x}{2}\right)$

The substitution $t = \tan\left(\frac{x}{2}\right)$ gives:

$\tan x = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}\,dt$

To show this recall the trignometric identity:

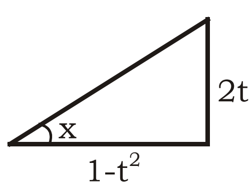
$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Then, substituting $A = \frac{x}{2}$

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$

$$\therefore \tan x = \frac{2t}{1-t^2}$$

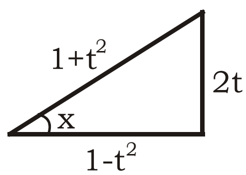
So, if we interpret *x* as an angle, this gives us the triangle:



The hypotenuse of this triangle is found by Pythagoras's Theorem:

$$\text{hypotenuse} = \sqrt{(2t)^2 + (1+t^2)^2}$$
$$= \sqrt{4t^2 + 1 - 2t^2 + t^4}$$
$$= \sqrt{t^4 + 2t + 1}$$
$$= \sqrt{(t^2+1)^2}$$
$$= t^2 + 1$$

Hence,



Whence

$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$

Finally, if $t = \tan\left(\frac{x}{2}\right)$

then $x = 2\tan^{-1} t$

$$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$$
$$\therefore dx = \frac{2}{1+t^2}\ dt$$

Solving integrals involving trigonometric functions

Definite and indefinite integrals involving trigonometric functions can sometimes be evaluated by use of the substitution $t = \tan\left(\frac{x}{2}\right)$.

Example (1)

Evaluate $\int \text{cosec } x\ dx$

Solution

$$\int \text{cosec } x\ dx = \int \frac{1}{\sin x}\ dx = \int \frac{1}{\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\ dt = \int \frac{1}{t}\ dt = \ln t + c = \ln\left(\tan\left(\frac{x}{2}\right)\right) + c$$

Example (2)

Find the integral of sec x

<u>Solution</u>

$$\int \sec x \; dx = \int \frac{1}{\cos x} \; dx$$

$$= \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2} \; dt$$

$$= \int \frac{2}{1-t^2} \; dt$$

$$= \int \frac{1}{1+t} \; dt + \int \frac{1}{1-t} \; dt \quad \text{[By decomposition into partial fractions.]}$$

$$= \ln|1+t| - \ln|1-t| + c = \ln\left|\frac{1+t}{1-t}\right| + c$$

It can be further shown, by use of trignometric identities, that

$\frac{1+t}{1-t} = \sec x + \tan x$.

Hence,

$$\int \sec x \; dx = \ln(\sec x + \tan x) + c$$

Example (3)

$$\int \frac{\sin x}{1+\cos x} \; dx = \int \frac{\left(\frac{2t}{1+t^2}\right)}{1+\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2} \; dt = \int \frac{2t}{(1+t^2)^2} \times \frac{2(1+t^2)}{(1+t^2+1-t^2)} \; dt = \int \frac{2t}{1+t^2} \; dt$$

$$= \ln(1+t^2) + c = \ln\left(1+\tan^2\left(\frac{x}{2}\right)\right) + c = \ln\left(\sec^2\left(\frac{x}{2}\right)\right) + c$$

The substitution $t = \tan x$

Another substitution that facilitates the integration of some trigonometric functions is $t = \tan x$

If $t = \tan x$ then $x = \tan^{-1} t$

$\therefore \dfrac{dx}{dt} = \dfrac{1}{1+t^2}$

So, $dx = \dfrac{1}{1+t^2}\ dt$

Also, since $\tan^2 x + 1 = \sec^2 x$

then, $\sec^2 x = 1 + t^2$

Hence, the set of substitutions is:

$t = \tan x$

$\sec^2 x = 1 + t^2$

$dx = \dfrac{1}{1+t^2}\ dt$

Example (4)

$$\int \frac{1}{1+\sin^2 x}\ dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\ dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\ dx$$

$$= \int \frac{1+t^2}{\left(1+t^2+t^2\right)} \times \frac{1}{\left(1+t^2\right)}\ dt$$

$$= \int \frac{1}{1+2t^2}\ dt$$

$$= \int \frac{1}{1+\left(\left(\sqrt{2}\right)t\right)^2}\ dt = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\ t\right) + c = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\ \tan x\right) + c$$

Example (5)

$$\int \frac{1}{1+\cos^2 x}\ dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}}\ dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 1} \; dx$$

$$= \int \frac{1+t^2}{\left(1+t^2+1\right)} \times \frac{1}{\left(1+t^2\right)} \; dt$$

$$= \int \frac{1}{2+t^2} \; dt$$

$$= \int \frac{1}{2\left(1+\frac{t^2}{2}\right)} \; dt$$

$$= \frac{1}{2}\int \frac{1}{1+\left(\frac{t}{\sqrt{2}}\right)^2} \; dt = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c$$

Example (6)

$$\int \frac{2\tan x}{\cos 2x} \; dx = \int \frac{2\tan x}{2\cos^2 x - 1} \; dx$$

$$= \int \frac{2\tan x}{\left(\frac{2}{\sec^2 x}\right) - 1} \; dx$$

$$= \int \frac{2\tan x \sec^2 x}{2 - \sec^2 x} \; dx$$

$$= \int \frac{2t\left(1+t^2\right)}{2-\left(1+t^2\right)} \times \frac{1}{1+t^2} \; dt$$

$$= \int \frac{2t}{1-t^2} \; dt = -\ln\left|1-t^2\right| + c = -\ln\left|1-\tan^2 x\right| + c$$