Integration by Substitution

Prerequisites

You should be familiar with direct integration. Direct integration essentially employs the method of trial and error to find an integral.

Example (1)

Use direct integration to find $\int \frac{x}{\sqrt{x-2}}\,dx$.

Solution

Try $\quad (x-2)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{d}{dx}(x-2)^{\frac{1}{2}} = \frac{1}{2}(x-2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-2}}$

Try $\quad x(x-2)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{d}{dx}x(x-2)^{\frac{1}{2}} = (x-2)^{\frac{1}{2}} + \frac{x}{2}(x-2)^{-\frac{1}{2}} = \frac{x-2}{\sqrt{x-2}} + \frac{\frac{x}{2}}{\sqrt{x-2}} = \frac{\frac{3x}{2}-2}{\sqrt{x-2}}$

Try $\quad \frac{d}{dx}\left\{\frac{2}{3}x(x-2)^{\frac{1}{2}} + \frac{8}{3}(x-2)^{\frac{1}{2}}\right\} = \frac{x-\frac{4}{3}}{\sqrt{x-2}} + \frac{\frac{4}{3}}{\sqrt{x-2}} = \frac{x}{\sqrt{x-2}} \qquad (\checkmark)$

Therefore

$$\int \frac{x}{\sqrt{x-2}}\,dx = \frac{2}{3}x(x-2)^{\frac{1}{2}} + \frac{8}{3}(x-2)^{\frac{1}{2}}$$

$$= \frac{2}{3}(x+4)\sqrt{x-2}$$

The method of substitution

This "nasty" example (1) above illustrates the obvious limitation of direct integration – namely, that it is uncertain and time consuming. In the final analysis anything that can be integrated can be integrated by direct integration – it is the ultimate technique to fall back on. However, it would be convenient if labour saving techniques could be introduced. Integration by the *method of substitution* is just such a labour saving device.

Example (2)

In example (2) we were asked to find $\int \frac{x}{\sqrt{x-2}}\,dx$.

(*a*) Let $u=\sqrt{x-2}$. Find $\frac{du}{dx}$.

(*b*) Substitute $u=\sqrt{x-2}$ into the answer you obtain in part (*a*) and find $\frac{dx}{du}$ in terms of *u*.

(*c*) Make *x* the subject of the equation $u=\sqrt{x-2}$

(*d*) Let $\int \frac{x}{\sqrt{x-2}}\,dx = \int \frac{x}{\sqrt{x-2}}\,dx \times \frac{du}{du}$

$$= \int \frac{x}{\sqrt{x-2}}\,\frac{dx}{du}\times du$$

Substitute for x, $\sqrt{x-2}$ and $\frac{du}{du}$ in this expression to obtain an integral in *u*.

(*e*) Integrate the result of part (*d*).

(*f*) By substituting $u=\sqrt{x-2}$ into the result obtained in part (*e*) find $\int \frac{x}{\sqrt{x-2}}\,dx$.

Solution

(*a*) $\frac{du}{dx} = \frac{1}{2}(x-2)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-2}}$

(*b*) $\frac{du}{dx} = \frac{1}{2u}$

$\frac{dx}{du} = 2u$

(*c*) $u=\sqrt{x-2}$

$u^2 = x-2$

$x = u^2+2$

(*d*) $\int \frac{x}{\sqrt{x-2}}\,dx = \int \frac{x}{\sqrt{x-2}}\,\frac{dx}{du}\times du$

$$= \int \frac{u^2+2}{u}\times 2u \times du$$

$$= 2\int \left(u^2+2\right)du$$

$$= 2\left(\frac{1}{3}u^3+2u\right)+c$$

$$= \frac{2}{3}\left(u^3+6u\right)$$

(*e*) $$\int \frac{x}{\sqrt{x-2}}\,dx = \frac{2}{3}\left(u^3 + 6u\right) + c$$

$$= \frac{2}{3}(x-2)^{\frac{3}{2}} + 6\sqrt{x-2} + c$$

$$= \frac{2}{3}(x+4)\sqrt{x-2} + c$$

This is the same as the result we obtained in example (1). Although the method used in example (2) also looks "nasty" in practice it represents a considerable labour saving device over the method of trial and error used in example (1). Firstly, provided you are given the substitution $u = \sqrt{x-2}$ the whole process is entirely mechanical, so no trial and error is involved. Even if you are not given the substitution, experience will suggest an appropriate substitution to try. Secondly, it is much easier to directly integrate $u^2 + 2$ than the original $\frac{x}{\sqrt{x-2}}$. Thus, the method of substitution turns a difficult direct integration into an easy one, and is entirely mechanical if you are given the substitution, and reduces the time spent on trial and error even if not.

The technique of integration by substitution

The essence of the technique of integration by substitution is to substitute an expression that involves *x* by something involving *u* and then to systematically replace everything with an *x* or *dx* in it by something with a *u* or *du* in it. When using integration by substitution we make the substitution

$$x = x(u)$$

into the integrand

$$y = \int f(x)\,dx$$

to obtain another integrand

$$\int f(x)\frac{dx}{du}\,du$$

that can be directly integrated. As the function $f(x) = f(x(u)) = f(u)$ can be expressed as a function of *u*, the integral $\int f(x)\frac{dx}{du}\,du$ becomes an integral in *u*. The purpose of the technique is to turn the difficult integral $y = \int f(x)\,dx$ into the easier integral $\int g(u)\,du$ where $g(u) = f(x(u))\frac{dx}{du}$.

Example (3)

Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{2x}{(2x-1)^2}\,dx$.

Solution

$u = 2x - 1 \qquad \Rightarrow \qquad 2x = u + 1$

$\dfrac{du}{dx} = 2 \qquad \Rightarrow \qquad dx = \dfrac{1}{2}du$

We will begin by finding the indefinite integral

$$\int \frac{2x}{(2x-1)^2}\,dx = \int \frac{u+1}{u^2} \times \frac{1}{2}du$$

$$= \int \frac{1}{2u} + \frac{1}{2u^2}du$$

$$= \frac{1}{2}\ln|2u| - \frac{1}{2}u^{-1} + c$$

$$= \frac{1}{2}\ln 2(2x-1) - \frac{1}{2}(2x-1)^{-1} + c$$

$$= \ln\sqrt{4x-2} - \frac{1}{2(2x-1)} + c$$

For the definite integral

$$\int_1^2 \frac{2x}{(2x-1)^2} = \left[\ln\sqrt{4x-2} - \frac{1}{2(2x-1)}\right]_1^2$$

$$= \left(\ln\left(\sqrt{6}\right) - \frac{1}{6}\right) - \left(\ln\sqrt{2} - \frac{1}{2}\right) = \ln\left(\frac{\sqrt{6}}{\sqrt{2}}\right) + \frac{1}{3} = \ln\left(\sqrt{3}\right) + \frac{1}{3}$$

In this example we were asked to evaluate a definite integral. We employed the following method

Method 1 for evaluating definite integrals using substitution.

Given $I = \int_a^b f(x)\,dx$

Make the substitution $u = g(x)$ and obtain the indefinite integral in the original variable *x*.

Evaluate the definite integral with the original limits.

There is another method that we could have employed.

Method 2 for evaluating definite integrals using substitution

Given $I = \int_a^b f(x)\,dx$

Make the substitution $u = g(x)$ and at the same time replace the limits by $g(a)$ and $g(b)$ to obtain the definite integral in terms of u

$I = \int_{g(a)}^{g(b)} f(x)\,dx$

Evaluate this definite integral in terms of *u*.

Both methods are equivalent and it is a matter of taste as to which to use. Method 2 is faster in practice and Method 1 is easier to understand. We will repeat example (3) using Method 2.

Example (3) continued

Use the substitution $u = 2x - 1$ to evaluate $\int_1^2 \frac{2x}{(2x-1)^2} dx$.

Solution

$u = 2x - 1 \quad \Rightarrow \quad 2x = u + 1$

$\frac{du}{dx} = 2 \quad \Rightarrow \quad dx = \frac{1}{2} du$

$x = 1 \quad \Rightarrow \quad u = 1$

$x = 2 \quad \Rightarrow \quad u = 3$

Then

$$\int_1^2 \frac{2x}{(2x-1)^2} dx = \int_1^3 \frac{u+1}{u^2} \times \frac{1}{2} du$$
$$= \int_1^3 \frac{1}{2u} + \frac{1}{2u^2} du$$
$$= \left[\frac{1}{2} \ln|2u| - \frac{1}{2} u^{-1} + c \right]_1^3$$
$$= \left(\frac{1}{2} \ln 6 - \frac{1}{6} \right) - \left(\frac{1}{2} \ln 2 - \frac{1}{2} \right)$$
$$= \ln\left(\sqrt{3}\right) + \frac{1}{3}$$

Example (4)

Evaluate $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \cos^4 x \, dx$

Solution

Method 1: Evaluating in the original variable with no change of limits.

$u = \cos x \Rightarrow \quad du = -\sin x \, dx$

$$\int \sin x \cdot \cos^4 x dx = -\int u^4 du = -\frac{u^5}{5} + c = -\frac{1}{5} \cos^5 x + c$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \cos^4 x \, dx = \frac{1}{5} \left[\cos^5 x \right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{1}{40} \sqrt{2}$$

Method 2: Evaluating in the substituted variable with a change of limits.

$u = \cos x \Rightarrow \qquad du = -\sin x\, dx$

$x = \frac{3\pi}{4} \qquad \Rightarrow \qquad u = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$x = \frac{\pi}{2} \qquad \Rightarrow \qquad u = \cos\frac{\pi}{2} = 0$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \times \cos^4 x dx = -\int_0^{-\frac{1}{\sqrt{2}}} u^4 du = -\left[\frac{u^5}{5}\right]_0^{-\frac{1}{\sqrt{2}}} = -\frac{1}{5}\left(-\frac{1}{4\sqrt{2}} - 0\right) = \frac{1}{40}\sqrt{2}$$

Theoretical justification

This section is optional.

In order to introduce you to the method of substitution as quickly as possible we conveniently ignored a step in our argument that in fact requires theoretical justification. This theoretical issue occurred in example (2) when we argued

$$\text{Let } \int \frac{x}{\sqrt{x-2}}\, dx = \int \frac{x}{\sqrt{x-2}}\, dx \times \frac{du}{du}$$

$$= \int \frac{x}{\sqrt{x-2}} \frac{dx}{du} \times du$$

In both these lines we are multiplying and dividing the expressions dx and du as if they were numbers or functions with numbers as their values. If you recall the theoretical foundation of integration, the symbols dx and du are derived from taking limits in the process of approximating an area. The obvious point, then, is that so far as we know they are not numbers and not functions either. So manipulating them in this way *may* be an error. Obviously we would not introduce this method or brush over the theoretical problem as we have if it was not possible to justify this method, as we shall now proceed to do. Furthermore, at a higher level it is possible to define the symbols dx and du to be functions, which means they can always be manipulated in this way. In advance of this justification let us make the following remarks.

Remarks

(1) The theoretical justification is important because it truly establishes a firm foundation for your knowledge of this subject. Knowing how and why things work is obviously much better than simply knowing that things work.

(2) However, as a technique, integration by substitution is very much learnt by practice, and the theory may make the technique seem much more complicated than it is.

Justification of integration by substitution

To prove

$y = \int f(x)\,dx$ and $x = x(u)$ then $\int f(x)\,dx = \int f(x)\,\frac{dx}{du}\,du$

<u>Proof</u>

Let (1) $y = \int f(x)\,dx$ (2) $x = x(u)$

By differentiating both sides of (1) $\frac{dy}{dx} = f(x)$

Since y is a function of x by (2) $x = x(u)$ it is a composite function. $y = y(x) = y\big(x(u)\big)$ Applying the chain rule for differentiation to this

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Since $\frac{dy}{dx} = f(x)$

$$f(x) = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = f(x)\frac{dx}{du}$$

Integrating both sides with respect to u

$$y = \int f(x)\frac{dx}{du}du$$

Substituting for y from (1)

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}du$$

Example (5)

Work your way through the proof above making the following changes of variable: x for y, t for x, u remains the same. The first line becomes

Prove $x = \int f(t)\,dt$ and $t = t(u)$ then $\int f(t)\,dt = \int f(t)\,\frac{dt}{du}\,du$.

Remark

In many applications the independent variable is often written as t (not x), which may stand for time, and the dependent variable is often written x (not y), which may represent position.

Solution

Let

(1) $x = \int f(t)\,dt$ (2) $t = t(u)$

By differentiating both sides of (1)

$$\frac{dx}{dt} = f(t)$$

From (2) $x(t) = x(t(u))$ is a composite function, so applying the chain rule to this

$$\frac{dx}{du} = \frac{dx}{dt} \times \frac{dt}{du}$$

Since $\frac{dx}{dt} = f(t)$

$$\frac{dx}{du} = f(t)\frac{dt}{du}$$

Integrating both sides with respect to u

$$x = \int f(t)\frac{dt}{du}du$$

Substituting for x from (1)

$$\int f(t)dt = \int f(t)\frac{dt}{du}du$$

Remark

The chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ expresses a fundamental idea about rates of change. This says that the rate of change of y with respect to x is equal to the rate of change of y with respect to u multiplied by the rate of change of u with respect to x. So when we have a composite function $y = u(x)$ we can simply multiply their corresponding rates of change. The notation $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ makes this clear. The method of substitution reverses this rule and applies it to integration as opposed to differentiation. So the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ is expressed as a relation between integrals $\int f(x)dx = \int f(x)\frac{dx}{du}du$ where $\frac{dx}{du}$ is the rate of change of x with respect to u.