Integrals involving inverse hyperbolic functions

Pre-requisites

You should be familiar with the following integrands that integrate to inverse trigonometric functions

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{dy}{dx} = \sin^{-1} x + c$$

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int -\frac{1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example (1)

Integrate $\int \frac{1}{\sqrt{16-x^2}}\,dx$

This is an application of the formula $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 4$

Hence

$$\int \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\left(\frac{x}{4}\right) + c$$

The purpose of this chapter is to extend the application of such formulae to other cases.

Standard forms for the integrals that integrate to inverse hyperbolic functions

The hyperbolic functions have the following logarithmic forms

$$\sinh^{-1} x = \ln\left|x + \sqrt{x^2+1}\right|$$

$$\cosh^{-1} x = \ln\left|x + \sqrt{x^2-1}\right|, \quad x \ge 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \quad |x| < 1$$

You should also be familiar from the study of hyperbolic functions of their derivatives

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

By reversing all of the above formulae we obtain the following formulae for integrals.

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c = \ln\left|x + \sqrt{x^2+1}\right| + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + c = \ln\left|x + \sqrt{x^2-1}\right| + c$$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + c = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

To obtain formulae of the form $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$ we hve

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c = \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right| + c = \ln\left|x + \sqrt{x^2+a^2}\right| + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + c = \ln\left|x + \sqrt{x^2-a^2}\right| + c \qquad x > 1$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c \qquad |x| < 1$$

Proofs

We have shown that $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$; then

$$\frac{d}{dx}\sinh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{1+\left(\frac{x}{a}\right)^2}} \times \frac{1}{a} \qquad \text{[By the chain rule]}$$

$$= \frac{1}{\sqrt{a^2\left(1+\frac{x^2}{a^2}\right)}}$$

$$= \frac{1}{\sqrt{a^2+x^2}}$$

We have already seen that $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$; then

$$\frac{d}{dx}\cosh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{\left(\frac{x}{a}\right)^2 - 1}} \times \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2\left(\frac{x^2}{a^2} - 1\right)}}$$

$$= \frac{1}{\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\left\{\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)\right\} = \frac{1}{a} \times \frac{1}{1-\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{1}{a^2 - x^2}$$

Applications of these formulae may be direct instances of substitution into them, or may require more insight and algebraic manipulation.

Example (2)

Find $\int \frac{1}{\sqrt{x^2+9}}\,dx$

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{1}{\sqrt{x^2+3^2}}\,dx$$

$$= \sinh^{-1}\left(\frac{x}{3}\right) + c$$

This is easily obtained by substituting a = 3 in $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

Completing the square

When the integrand involves an expression of the form $ax^2 + bx + c$ then complete the square and use a trigonometric substitution or otherwise to evaluate the integral.

Example (3)

Find $\int \frac{1}{x^2+2x+4}\,dx$

Completing the square

$x^2 + 2x + 4 = x^2 + 2x + 1 + 3 = (x+1)^2 + 3$

Then

$$\int \frac{1}{x^2+2x+4}\,dx = \int \frac{1}{(x+1)^2 + 3}\,dx$$

Let $u = x + 1 \Rightarrow du = dx$

Then

$$\int \frac{1}{x^2+2x+4}dx = \int \frac{1}{u^2+3}du$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

Example (4)

Find $\int \frac{1}{\sqrt{x^2+2x+4}}dx$

Completing the square as in the previous example

$$\int \frac{1}{\sqrt{(x+1)^2+3}}dx$$

As before let $u = x+1 \Rightarrow du = dx$

Then

$$\int \frac{1}{\sqrt{(x+1)^2+3}}dx = \int \frac{1}{\sqrt{u^2+3}}du$$
$$= \sinh^{-1}\left(\frac{u}{\sqrt{3}}\right)+c$$
$$= \sinh^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+c$$

Or in the logarithmic form

$$\int \frac{1}{\sqrt{(x+1)^2+3}}dx = \int \frac{1}{\sqrt{u^2+3}}du = \ln\left|u+\sqrt{u^2+3}\right|+c = \ln\left|x+1+\sqrt{x^2+2x+4}\right|$$

Example (5)

Find $\int \frac{1}{\sqrt{12+6x-x^2}}dx$

Completing the square

$$12+6x-x^2 = -\left(x^2-6x\right)+12$$
$$= -\left(x^2-6x+3^2-3^2\right)+12$$
$$= -\left(x-3\right)^2+21$$
$$= 21-\left(x-3\right)^2$$

Let $u = x-3 \Rightarrow du = dx$

Then

$$\int \frac{1}{\sqrt{12+6x-x^2}}\,dx = \int \frac{1}{\sqrt{21-(x-3)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{21-u^2}}\,du$$

$$= \sin^{-1}\left(\frac{u}{\sqrt{21}}\right) + c$$

$$= \sin^{-1}\left(\frac{x-3}{\sqrt{21}}\right) + c$$

Example (6)

Find $\int \frac{1}{x^2 + x - \frac{1}{2}}\,dx$

Completing the square on $x^2 + x + 1$

$$x^2 + x - \frac{1}{2} = x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \frac{1}{2} = \left(x + \frac{1}{2}\right)^2 - \frac{3}{4}$$

then

$$\int \frac{1}{x^2 + x - \frac{1}{2}}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 - \frac{3}{4}}\,dx$$

$$= -\int \frac{1}{\frac{3}{4} - \left(x+\frac{1}{2}\right)^2}\,dx$$

Let $u = x + \frac{1}{2} \qquad \Rightarrow \qquad du = dx$

$$-\int \frac{1}{\frac{3}{4} - \left(x+\frac{1}{2}\right)^2}\,dx = -\int \frac{1}{\frac{3}{4} - u^2}\,du$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \times \tanh^{-1}\left(\frac{u}{\sqrt{3}/2}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{2u}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Hyperbolic substitutions for the evaluation of integrals

You should be already familiar with the technique of integration by substitution. In this section we observe that sometimes an integral can be found by means of a hyperbolic substitution.

Example (7)

Find $\int \sqrt{x^2 - a^2}\, dx$ using the substitution $x = a\cosh u$

We have $x = a\cosh u$ then $x^2 = a^2\cosh^2 u$ so

$$\begin{aligned}\sqrt{x^2 - a^2} &= \sqrt{a^2\cosh^2 u - a^2}\\ &= a\sqrt{\cosh^2 u - 1} \qquad \left[\text{Since } \cosh^2 u - \sinh^2 u = 1\right]\\ &= a\sqrt{\sinh^2 u}\\ &= a\sinh u\end{aligned}$$

Also $\dfrac{dx}{du} = a\sinh u \qquad \Rightarrow \qquad dx = a\sinh u \cdot du$

Hence

$$\begin{aligned}\int \sqrt{x^2 - a^2}\,dx &= \int a\sinh u \cdot a\sinh u \cdot du\\ &= a^2\int \sinh^2 u \cdot du\\ &= a^2\int \frac{1}{2}(\cosh 2u - 1)\,du \qquad \left[\cosh 2u = 1 + 2\sinh^2 u\right]\\ &= \frac{a^2}{2}\left(\frac{\sinh 2u}{2} - u\right)\\ &= \frac{a^2}{4}\sinh 2u - \frac{a^2}{2}u\\ &= \frac{a^2}{4}(2\sinh u\cosh u) - \frac{a^2}{2}u\\ &= \frac{1}{2}(a\sinh u)(a\cosh u) - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) \qquad \left[\sinh 2u = 2\sinh u\cosh u\right]\\ &= \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)\end{aligned}$$

We can also put this into logarithmic form

$$\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\left(\ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\right)$$

Example (8)

Find $\int \dfrac{1}{\sqrt{x^2 - a^2}}\,dx$ using the substitution $x = a\cosh u$

As before we have $\sqrt{x^2 - a^2} = a\sinh u$ and $dx = a\sinh u \cdot du$

Hence

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \int \frac{1}{a\sinh u} a\sinh u \cdot du$$
$$= \int du$$
$$= u$$
$$= \cosh^{-1}\left(\frac{x}{a}\right)$$

The form $\dfrac{1}{a + b\cos x}$

To find an integral of the form $\int \frac{1}{a + b\cos x} dx$ we require the substitution $t = \tan\left(\frac{x}{2}\right)$. The student should already be aware that with this substitution we obtain $\tan x = \frac{2t}{1 - t^2}$, $\sin x = \frac{2t}{1 + t^2}$, $\cos x = \frac{1 - t^2}{1 + t^2}$. Furthermore, if $t = \tan\left(\frac{x}{2}\right)$ then

$$x = 2\tan^{-1} t$$
$$dx = \frac{2}{1 + t^2}\, dt$$

When this substitution is made a form arises that integrates to either an inverse trigonometric or an inverse hyperbolic function.

Example (9)

Find $\int \frac{1}{1 + 2\cos x} dx$

Let $t = \tan\left(\frac{x}{2}\right)$ whence $\cos x = \frac{1 - t^2}{1 + t^2}$, $dx = \frac{2}{1 + t^2}\, dt$ and on substituting into $\int \frac{1}{1 + 2\cos x} dx$ we obtain

$$\int \frac{1}{1 + 2\cos x} dx = \int \frac{1}{1 + 2\left(\frac{1 - t^2}{1 + t^2}\right)} \times \frac{2}{1 + t^2}\, dt$$
$$= \int \frac{2}{3 - t^2} dt$$
$$= \frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{t}{\sqrt{3}}\right)$$
$$= \frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{\tan\left(x/2\right)}{\sqrt{3}}\right)$$