MELAMPUS

On the Continuum

www.melampus.name

On the Continuum

Contents

On the Continuum

1. The problem of the continuum

In first-order set theory real numbers are identified with subsets of the first infinite ordinal, ω. We call the collection of all real numbers, denoted $\mathbb{R}$, the **arithmetical continuum**, or continuum for short. We also call it **the real line**. The continuum is usually identified with the set of all subsets of ω, which is the powerset of ω, denoted $\mathbf{P}(\omega)$, and is also the set of all functions, $\omega \to \mathbf{2} = \{0,1\}$. What this means is that every real number has a unique binary expansion [See section 18.2]. Hence, the powerset is also isomorphic to the infinite product of the discrete space, $\mathbf{2} = \{0,1\}$, and we write, $\mathbf{P}(\omega) = \mathbf{2}^{\omega} = \{0,1\}^{\omega}$. This is called the **Cantor set**. Any space isomorphic to the Cantor set is called **Cantor space**. Cantor used an anti-diagonalisation argument[1] to demonstrate that the cardinality of $\mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$ is greater than that of the cardinality of ω. Denoting the cardinality of ω by $\text{card}(\omega) = |\omega| = \aleph_0$,[2] the cardinality of the continuum by $\mathfrak{c}$, and the cardinality of $\mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$ by $2^{\aleph_0}$, this gives $|\mathbb{R}| = \mathfrak{c} = 2^{\aleph_0} > \aleph_0$. **Hartog's theorem** establishes that there is a succession of cardinals, $\aleph_0, \aleph_1, \aleph_2, \ldots$, though the structure of the upper-end of this succession requires further axioms to be determined. The question arises, where in the succession of cardinals does the cardinal $\mathfrak{c} = 2^{\aleph_0}$ come? Cantor conjectured the **Continuum Hypothesis**: $2^{\aleph_0} = \aleph_1$. I shall call the determination of the size, $\text{card}(\mathbf{2}^{\omega})$, of the continuum: **the problem of the continuum**.

Since the discovery of forcing arguments by Cohen [1966] we have come to realise that there are many ways to answer the question, what is $\text{card}(\mathbf{2}^{\omega})$? Each way depends upon a selection of forcing arguments and associated additional axioms. This has led to relative consistency results. Taking **ZFC** (Zermelo-Frankel set theory with Choice) as the standard axiomatisation of first-order set theory, it is clear that ZFC is consistent with almost any determination of the size of the continuum.

[1] Cantor introduced two arguments: the first to establish when two sets are equinumerous, which I designate **diagonalisation**; the second to establish when one infinite set is not equinumerous to the other, which I designate **anti-diagonalisation**. It is the latter that establishes that the cardinality of the continuum is greater than that of the natural numbers.

[2] The symbol $|X|$ for the cardinality of a set X is standard; but sometimes we wish to particularly emphasise that we are dealing with the cardinality of the set as opposed, say, to its length or measure, so we have $\text{card}(X)$ as well.

Thus in ZFC the size of the continuum, $\text{card}\left(2^{\omega}\right)$, is under-determined. This has the immediate air of paradox, because a representing set of the continuum such as the Cantor set, $\mathbf{P}(\omega) = \mathbf{2}^{\omega} = \{0,1\}^{\omega}$, should be an unambiguous structure completely determined "up to isomorphism". What the relative consistency results tell us is that ZFC is a "weak" theory – it is a theory capable of many models. It follows that as a model of the arithmetical continuum the Cantor set represents not one but a plethora of models. But since the Cantor set "ought" to be an unambiguous structure, identical to others up to isomorphism, this situation requires explanation. So the problem of the continuum comprises not only the determination of its cardinality but more fundamentally a determination of an appropriate structure for the continuum, one that makes the structure of the Cantor set, $\mathbf{2}^{\omega}$, as a representing set of the continuum unambiguous.

The set theoretical view of ω is that it is the least infinite ordinal. This makes it a transitive set whose members are the finite ordinals. Since the Cantor set is the power set of ω this means that ω must be related in some intimate way to the structure of the continuum. Since $\text{card}\left(\mathbf{2}^{\omega}\right)$ is capable of many solutions, it follows that the role of ω is also ambiguous: there must be as many ways to relate ω to the structure of the continuum as there are solutions to $2^{\aleph_0} = \aleph_{\alpha}$.

In analysis we use the Axiom of Completeness. For example, standard results in analysis, such as Rolle's theorem or Cauchy's mean-value theorem depend on it. The Axiom of Completeness was formulated in the 1890s by Dedekind, Weierstrass, Borel, Cantor and others. It is at this point that we make a striking discovery: while the Axiom of Completeness is the principle and distinctive tool of analysis and provides the greatest insight into the structure of the real line, I can find no explicit use of it or exploration of its consequences in the literature discussing the continuum from 1900 onwards. A theory is said to be **first-order** if whenever we quantify in the theory, that is refer to collections using the term "all", the objects that we collect are sets or individuals. A theory is said to be **second-order** if we allow ourselves to quantify over properties and functions, or equivalently, over sets of sets. **First-order logic** is concerned with deductions between first-order expressions; and **second-order logic** with deductions between expressions that are second-order. It is not possible to give a first-order formulation of the Axiom of Completeness

In this monograph, I shall demonstrate that the Axiom of Completeness entails the Continuum Hypothesis and provides an unambiguous determination of the structure of the real line. I shall also explain the role of ω in that determination. The principle tools, whose meanings shall be subsequently defined, are: -

1. The skeleton of the line;
2. The one-point compactification of the skeleton;
3. The Mahler classification of transcendental numbers.

All of these principles depend upon the Axiom of Completeness together with the Dirichlet pigeonhole principle, which is implicit in all discussions. I shall employ ZFC as a language, drawing on its axioms as required. In fact, not all of ZFC is required, since the full scope of the power set axiom is not used. However, since this is an exercise in second-order logic, the restrictions of working in ZFC, which cause set theorists much labour, shall not apply here.

It is important to stress from the outset that the result presented here is a conditional one. What shall be established is that the Axiom of Completeness and the Dirichlet Pigeonhole principle together entail the Continuum Hypothesis. On the other hand, this does come at a price – for example one consequence of this model is the existence of non-measurable sets. Furthermore, it exposes all the more clearly the possibility of alternative interpretations of the continuum. In the long run I suggest that the determination of the structure of the continuum and its cardinality will become empirical questions.

The parallel postulate of Euclid would be a very good illustration of how our interpretation of an axiom may undergo change. Consider an axiom, Δ. At the first stage Δ appears to be grounded in intuition and self-evident. No alternative to the axiom is even conceivable. At the second stage, notwithstanding the apparent self-evident aspect of Δ, doubts about its status emerge, and a research program starts with a view to deducing Δ from other axioms that are still held to be self-evident. It is still not possible to imagine an alternative to *A*, and no model of $\neg A$ is as yet available; nonetheless, its status as self-evident truth grounded in intuition has by this stage been called into question. At stage three several models of $\neg\Delta$ emerge. It is then acknowledged that Δ is not self-evident and that $\neg\Delta$ is conceivable. Finally, by stage four, whether Δ is true or false has become an empirical question. Alternative empirical theories, some based on Δ and others based on $\neg\Delta$, emerge, and these become subject to empirical confirmation.

With the Continuum Hypothesis we have not yet reached the fourth stage of this process; and we are still stuck confusedly in the third stage.

2. The potential and actual infinite

2.1 The potential infinite and the Archimedean property

The distinction between the potential and actual infinite is an important part of the conceptual background to an understanding of the continuum. Like the second-order Axiom of Completeness, this distinction has been largely ignored for more than a century.

Aristotle [2012/c 350 BC] in his *Physics* introduced the distinction. The **potential infinite** is illustrated by the inexhaustibility of counting; for no mater how large a number we have reached it is always possible to count to a higher one merely by adding one more. In the actual infinite we conceptualise the entire process of counting as a **completed totality**.

The potential infinite is symbolised by ∞, which represents the inexhaustibility of counting. The natural numbers are 1, 2, 3, ...[3] Here the dots also indicate the potential infinite. When we wish to talk of the collection of all natural numbers we represent this by $\mathbb{N}$. This collection is a **potential infinity.**

In the **actual infinite** we have a conception of a completed collection of infinite objects – usually points or numbers – as given actually in its entirety. The theory of actually infinite collections is *set theory*, which employs many symbols for such objects. The least infinite collection is denoted by ω. This is the collection of **finite ordinal numbers** – which are sets ordered in succession.

Whereas the collection of all finite ordinals, ω, is bounded above, the collection of natural numbers, $\mathbb{N}$, is not bounded above. The property of not being bound above is equivalent to the **Archimedean property**.

2.1.1 Archimedean property

If *a* and *b* are any particular integers, then there exists a positive integer *n* such that $na \geq b$. (Burton [1976] p.2) This implies that $\mathbb{N}$ is not bounded above.

The ordinal ω is conceptualised as another ordinal following in succession after all the finite ordinals. But, given that all set theorists regard the collections $\mathbb{N}$ and ω as one and the same set, casting doubt on this identification may seem surprising and unacceptable. Nonetheless, the validity and importance of the distinction will emerge as the argument unfolds. The difference between $\mathbb{N}$ and ω is encapsulated by the following result: -

2.1.2 Lemma

For ω the following statements are equivalent: -

1. ω is a limit ordinal
2. $(\forall n)(n < \omega \supset n+1 < \omega)$
3. $\omega = \sup_{n<\omega} n$

(Proof in Potter [2004] p. 181)[4]

In the second of these statements: we cannot substitute $\mathbb{N}$ for ω; the expression $< \mathbb{N}$ is meaningless, and $\mathbb{N}$ is the collection of natural numbers not its least upper bound. On the other hand ω is the least upper bound (supremum) of all ordinals. Thus, if $\omega = \mathbb{N}$ we must allow the collection of all natural numbers, $\mathbb{N}$, to be bounded, contradicting the Archimedean

[3] The omission of 0 from this list is simply pragmatic. We use 0 for the zero of a lattice, 0 may sometimes appear as synonym of the null set, 0 as a point in the closed interval $[0]$, so to avoid confusion we are taking 0 out of the natural numbers.

[4] I have slightly adapted the theorem in Potter which is for all limit ordinals. Only ω concerns us here.

property. We must either dispense with the Archimedean property or drop the commonplace identification of $\omega = \mathbb{N}$.

> … there is a least limit ordinal, which is called ω ("omega"). The members of ω are called **finite** ordinals or **natural numbers**. In other words, to a set theorist $\omega = \mathbb{N}$. (Wolf [2005] p. 83) [5]

On the other hand, if we allow the Completeness Axiom, then we must draw the distinction. This is because we can derive the Archimedean property of the natural numbers from the Completeness Axiom.

2.1.3 Analytic proof of the Archimedean property from the Completeness Axiom

Suppose $\mathbb{N}$ is bounded above. Then by the completeness axiom there exists a unique real number *u*, such that $u = \sup \mathbb{N}$. For any number $n \in \mathbb{N}$ the number $n+1 \in \mathbb{N}$, hence $n+1 \leq u$ and $n \leq u-1$. This is true for all $n \in \mathbb{N}$, hence $u-1$ is an upper bound for $\mathbb{N}$. This contradicts the uniqueness of *u*, so $\mathbb{N}$ cannot be bounded above.[6]

Contrary to appearances, it turns out that the distinction between $\mathbb{N}$ and ω is already implicit in set theory, which does have devices to deal with potential infinities, even if it does not "officially" recognise them. One such device is the expression $< \omega$; the distinction between $< \omega$ and ω is vital to the discourse on the continuum, yet the two sets would *appear* to have the same enumeration: -

$$< \omega = \{0,1,2,...\} \qquad \omega = \{0,1,2,...\}$$

We also meet the potential infinite in the concepts of an open set or interval, and in the concept of cofinality. In first-order set theory where we implicitly draw a distinction between the potential and actual infinite, we use the symbol, $< \omega$.

---

[5] For another statement of this kind: Ordinals are the order types of well-ordered sets. They are the infinite analogues of the natural numbers, and in many respects they behave like the latter ones. In fact, the finite ordinals are the natural numbers, and hence the transfinite class of ordinals can be considered as an endless continuation of the sequence of natural numbers. (Komjáth and Totik [2000] p. 37) [My underlining.]

[6] It might be objected that in set theory, the unique real number that is the supremum of $\mathbb{N}$ is $\omega = \sup \mathbb{N}$ and that the expression, $\omega - 1$, is meaningless. However, in the argument above *u* is a real number, so we may subtract from it. It is not assumed that the supremum is ω.

3. The phenomenological and the arithmetical continuum

3.1 The phenomenological continuum

The phenomenological continuum of space and time that was described by Aristotle in his *Physics* is not used in modern mathematics. However, it is the starting point for our intuitions regarding the continuum. This concept was refined by Kant in *The Critique of Pure Reason.*

The phenomenological continuum is the continuum that we meet in experience. It is the continuum that is presented directly to our own eyes and mind. Key features of this concept are: -

1. Points are dimensionless. No continuum is composed of points. Points act as boundaries or limits only. They are fictional (that is "ideal") elements.[7]
2. Space is composed only of space; subdivision of the continuum generates only another continuum.[8]
3. Subdivision of the continuum may carry on indefinitely. Thus, space is potentially infinite. Space is never actually infinitely divisible.[9]

---

[7] "Now if the terms 'continuous', 'in contact', and 'in succession' are understood as defined above things being 'continuous' if their extremities are one, 'in contact' if their extremities are together, and 'in succession' if there is nothing of their own kind intermediate between them – nothing that is continuous can be composed 'of indivisibles': e.g. a line cannot be composed of points, the line being continuous and the point indivisible. For the extremities of two points can neither be one (since of an indivisible there can be no extremity as distinct from some other part) nor together (since that which has no parts can have no extremity, the extremity and thing of which it is the extremity being distinct)." Aristotle [2012 / c 350 BC Book VI, §1]

[8] "Nor, again, can a point be in succession to a point or a moment to a moment in such a way that length can be composed of points or time of moments: for things are in succession if there is nothing of their own kind intermediate between them, whereas that which is intermediate between points is always a line and that which is intermediate between moments is always a period of time." Aristotle [2012 / c 350 BC Book VI, §1]

"The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are *quanta continua*, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space and a time. Space therefore consists solely of spaces, time solely of times. Points and instants are only limits, that is, mere positions which limit space and time. But positions always presuppose the intuitions which they limit or are intended to limit; and out of mere positions, viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed. ... Kant, Immanuel [1982 / 1781] Critique of Pure Reason. Part 2. Transcendental Logic. First Division. Chapter II. § 3.2 Anticipations of Perception. Translated by Norman Kemp Smith, First edition 1929. (Reprinted 1982) p. 204]

[9] "We must keep in mind that the word 'is' means either what potentially is or what fully is." [Book III, §6] "... magnitude is not actually infinite. But by division it is infinite. ... The alternative then remains that the infinite has a potential existence.")." Aristotle [2012 / c 350 BC] Book III, §6.

Greek science adopted the phenomenological continuum as a basis of science. Historically, this prevented the emergence of the calculus, which requires the actual infinite. Greek mathematicians adopted other methods in order to obtain practical formulas for integrals – specifically, the method of exhaustion.

3.2 The "necessity" in analysis for the actual infinite

$\mathbb{R}$ is the complete set of all ratios of measure of part to whole. Let I_1 represent a whole interval, and I_2 a part. Then $\frac{I_1}{I_2} \in \mathbb{R}$. If the ratio $\frac{I_1}{I_2} = \frac{dy}{dx}$ is irrational and is deemed not to belong to the continuum $\left(\frac{I_1}{I_2} \in \mathbb{R}\right)$, then the continuum is incomplete.

Consider two distinct sequences that converge on the "same irrational number", say $\sqrt{2}$. But if we do not have a notion of a two sequences being actually infinite, then it is strictly not possible to talk of them as converging to the same limit, for $\sqrt{2}$ does not exist.

Historically, set theory developed out of analysis: sets began their existence as sequences. Let us say $\sqrt{2}$ is the name of such a sequence. Then $\sqrt{2}$ does not name the limit but only the sequence. It is an inductive rule for systematically generating ratios and is connected to the process of counting and the potential infinite. The sequence cannot be grasped without the rule, which is its concept (or intension). So, at this stage, $\sqrt{2}$ is an intensional object. With the intensional concept of sequence, each sequence is different, *so there are as many* $\sqrt{2}$ *as there are sequences that approximate to it.* In order to prevent this situation whereby different sequences with the same limit are different numbers, we must introduce a unique number that is the identical limit of these differing sequences. This transforms the intensional notion of sequence into the extensional notion of an equivalence class of ordered sets and into an actual infinity. Cauchy added $\sqrt{2}$ not as the name of a sequence but as the name of the limit of a sequence. Under this conception, different sequences can have the same limit.

But even this is not enough, for the set of all definable sequences is countably infinite. What about limits that cannot be identified by a rule? We wish to talk of all sequences and all limits independently of whether we can construct the sequence or not. So we extend our concept of number to include the concept of a set of all numbers, which are limits, regardless

---

"The infinite, then, exists in no other way, but in this way it does exist, potentially and by reduction." Aristotle [2012 / c 350 BC] Book III, §6.

"But in the direction of largeness it is always possible to think of a larger number: for the number of times a magnitude can be bisected is infinite. Hence this infinite is potential, never actual: the number of parts that can be taken always surpasses any assigned number. But this number is not separable from the process of bisection, and its infinity is not a permanent actuality but consists in a process of coming to be, like time and the number of time." Aristotle [2012 / c 350 BC] Book III, §7.

of whether we can construct them or not. This is what happens in Dirichlet's characteristic function: -

$$\chi(x) = \begin{cases} 1 \text{ iff } x \text{ is rational} \\ 0 \text{ iff } x \text{ is irrational} \end{cases}$$

This is based on the totality of all real numbers, regardless of how they are constructed.

The completeness axiom asserts the independent existence of limits, independently of the sequences that approximate to them, so explicitly allows for quantification over limits as now real numbers. From this arises the concept of the arithmetical continuum.

The term "necessity" in the heading of this subsection is in scare quotes. The actual infinite is part of a system of explanation whose validity ultimately rests on empirical evidence. Even if we allow that there are some necessary truths grounded in intuition, the concept of the actual infinite could not be justified in that way. It is necessary because we deem it to be so. Nonetheless, this does not make it a matter of convention either, so a constructivist, formalist or strict finitist solution to these problems is not even hinted at here.

3.3 The arithmetical continuum

This is illustrated by the following quotation from G.H. Hardy: -

> The aggregate of all real numbers, rational and irrational, is called the *arithmetical continuum.*
>
> It is convenient to suppose that the straight line … is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others. The points of the line, the aggregate of which may be said to constitute the *linear continuum*, then supply us with a convenient image of the arithmetical continuum. Hardy, G.H. [1908 / 1967] Chapter 1. §15.

This conception contradicts the phenomenological continuum of Aristotle. It states that a line is composed of points.

3.4 Fundamental problem of the arithmetical continuum

Assuming the points have no size whatsoever, then the line is an infinite collection of objects that have no size. Can a mere collection of points, which are boundaries, limits or extremities, ever become extended in space? This question poses the fundamental paradox of the arithmetical continuum.

The real line comprises $\mathfrak{c} = \text{card}(\mathbb{R})$ of uncountably infinite points. Any collection of countably infinite points $(\aleph_0)$ has zero measure. There exist countable collections of open sets (intervals) with positive measure. There exist collections of continuum many points, $\mathfrak{c}$, whose measure is zero. These are perfect sets isomorphic to the Cantor set $SVC(3) = \mathbf{2}^{\omega}$. Yet the Cantor set $SVC(3) = \mathbf{2}^{\omega}$ is also a representing set for the continuum, $\mathbb{R}$. We cannot prima facie derive measure from cardinality. For example, we cannot adopt as a rule: all sets of points of cardinality continuum have positive measure, and all sets of cardinality less than continuum have zero measure. Such a rule is inconsistent.

3.5 Intervals and the order topology

In point set theory an interval is defined by reference to the order topology.

3.5.1 Definition, open interval

Let $\langle X,<\rangle$ be an ordered set. An open interval of X is a set which is of the form $(u,v) = \{z \in X : u < z < v\}$ where $u < v$. Levy, Azriel [2002] p.200.

In this definition, there is no reference to either continuity or connectedness. So an interval in point set theory does not have to be connected: $\left(\frac{1}{2},\frac{3}{4}\right)$ is an interval in $\mathbb{Q}$ but is not connected and not continuous; $\left(\frac{1}{2},\frac{3}{4}\right)$ is an interval in $\mathbb{R}$ but is connected and continuous with positive measure $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Intervals may be defined, but they do not carry with them a "primitive" notion of extension. Intervals may be mere collections of discrete points. When does an interval have a positive measure, and when does it not?

4. Axiom of Completeness

4.1 Equivalent forms of the Axiom of Completeness

The following five statements are equivalent versions of the Completeness Axiom: -

1. **Dedekind completeness axiom**

   Let the set of all real numbers, $\mathbb{R}$, be divided into two sets *L*, *R* such that every member *l* of L is less than every member *r* of R, where neither *L* nor *R* is empty. Then there exists a number ξ such that every number less than ξ belongs to L and every number greater than ξ belongs to R. The number ξ is said to divide the set $\mathbb{R}$. The number ξ may be a member of either *L* or *R*. If it is a member of L then it is the greatest member of L; if it is a member of *R* then it is the least member of R.[10]

[10] More succinctly: Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound in the set. The least upper bound is called its *supremum.*

2. **Bolzano-Weierstrass theorem**

   Every infinite bounded subset has a limit point in the set.

3. **Cantor's nested interval principle**

   Given any nested sequence of closed intervals in $\mathbb{R}$,

   $[a_1,b_1]\supseteq[a_2,b_2]\supseteq\ldots\supseteq[a_n,b_n]\supseteq\ldots$,

   there is at least one real number contained in all these intervals: -

   $\bigcap_{n=1}^{\infty}[a_n,b_n]\neq\varnothing$

4. **Cauchy convergence criterion**

   Let *S* by a non-empty subset of $\mathbb{R}$. Every Cauchy sequence on *S* converges to a real point in *S*.

5. **Heine-Borel theorem of real analysis**

   Let *X* be a closed, bounded set on the real line $\mathbb{R}$. Then every collection of open subsets of $\mathbb{R}$ whose union contains *X* has a finite subclass whose union also contains *X*.

The Completeness Axiom is irreducibly a statement of second-order logic. As Woff remarks: -

> "… the completeness property – that every set of reals with an upper bound has a least upper bound – is unavoidably second order…" Wolf, Robert S [2005] p.43.

It is irreducibly second order because it refers to all subsets of the domain, $\mathbb{R}$, and there is no way to state it without such a reference. The Heine-Borel theorem was deduced from the Dedekind Completeness Axiom before it was realised that the two statements are equivalent. It is important to examine the proof of the Heine-Borel theorem.

4.1.1 Proof summary of the Heine-Borel theorem

Let $[a,b]$ be a closed interval in $\mathbb{R}$. Assume it has a cover, possibly infinite. Then there is a neighbourhood $U_{b,\varepsilon}=\{x\in[a,b]:b-x<\varepsilon,\text{for all }\varepsilon\in\mathbb{R}\}$ which is a member of some cover for $[a,b]$. Removing this neighbourhood from $[a,b]$ we obtain the interval $[a,b)=[a,b-\varepsilon)$. By defining suprema on finite subsets of this interval, it follows from the Completeness Axiom that there is a finite cover, a subsequence, Σ, that covers $[a,b-\varepsilon)$. Then $\Sigma\cup U_{b,\varepsilon}$ is a finite cover for $[a,b]$.

The aim here is to show the close relationship between the Heine-Borel theorem and the one-point compactification of $[a,b)$.

4.1.2 Definition, Alexandroff 1-point compactification

The addition of a neighbourhood of b to the half-open interval $[a,b)$ shall be called the **Alexandroff 1-point compactification** of $[a,b)$.[11]

Thus, every closed interval, $[a,b]$, in the arithmetical continuum, $\mathbb{R}$, has a one-point compactification.

4.2 Separability

Suslin characterized the real line (arithmetical continuum) as the unique separable, complete, linear order with neither end points nor isolated points.[12]

4.2.1. Definition, separable

A separable space has a countable dense subset.

4.2.2 Definition, Hausdorff space

A topological space, *T*, is Hausdorff if given any two distinct points $x, y \in T$ there exist distinct disjoint open subsets *U*, *V* of *T* containing *x* and *y* respectively.

The arithmetical continuum, $\mathbb{R}$, is both separable and Hausdorff, and on $\mathbb{R}$ these represent equivalent notions. Separability is an essential component of the concept of the arithmetical continuum because it expresses the manner in which real numbers are constructed out of the sequences of rational numbers that approximate them. Real numbers do not appear "out of nowhere", but emerge on closure of their rational dense subset. By Cantor's anti-diagonalisation argument, the number of reals generated in this fashion is of uncountable cardinality $\mathfrak{c}$. The Hausdorff condition requires different convergent sequences to converge to unique points: -

4.2.3 Proposition

In a Hausdorff space any given convergent sequence has a unique limit.

All three properties, 1-point compactification, separability and the Hausdorff condition are consequences of the Axiom of Completeness. It is also worth remarking that the Hausdorff condition is also a topological invariant, meaning that it is preserved by homeomorphisms.

[11] The following is the definition given in Givant, Steven and Halmos, Paul [2009]: "If *Y* is a topological space, and if *X* is obtained from *Y* by adjoining a single point x_0, then the *one-point compactification* of *Y* is the set *X* with the following topology: the open subsets of *X* are defined to be the open subsets of *Y* and the complements in *X* of the closed compact subsets of *Y*." p. 344.

[12] Described in Stephrans, Juris [X]

4.3 On the difference between first-order and second-order theories

The approach of first-order set theory is to convert second-order axioms into first-order definitions.

The Axiom of Completeness makes an existential claim: if a sequence is convergent, then there exists a unique limit to that sequence. Not only does it call into existence the limit as a real number, it also calls into existence the collection of all those real numbers, the arithmetical continuum. Therefore, from a historical point of view at least, the arithmetical continuum and the Axiom of Completeness are one and the same. The arithmetical continuum is the topologically invariant structure that the Axiom of Completeness calls into existence.

In set theory no such existential claims are made. A structure is defined to have certain properties – for example, it is a space that is complete, separable, and so forth, but it is not asserted that the structure exists. Except for the simplest cases, first order axioms and definitions do not give categorical structures. This set-theoretic approach allows us to conceive of different structures, different spaces within which to do our physics, and in this manner it assists the expansion of the empirical method.

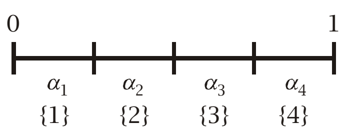
The arithmetical continuum is not given *a priori*; it is not, in the language of Descartes, manifested self-evidently by the clear light of reason. It is a hypothesis of science. Nonetheless, we need to appreciate the centrality of the Axiom of Completeness. The problem of the continuum arises because we are attempting to use first-order definitions to solve a problem that can only be solved by second-order axioms. First-order set theory is not categorical for the continuum.

5. Partitions and lattices

5.1 Finite partitions of the unit interval

The arithmetical continuum, denoted $\mathbb{R}$, is homeomorphic to any open interval: $\mathbb{R} \cong (a,b)$. However, it will be useful to take as our representative of the arithmetical continuum the closed unit interval: $\mathbb{I} = [0,1] \cong \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. We choose a closed interval because our main tool of analysis shall be the Axiom of Completeness in the form of the Heine-Borel theorem, which requires a compact interval (i.e one closed and bounded), and it makes sense to normalise this by taking its measure (length) to be $\mu([0,1]) = 1$.

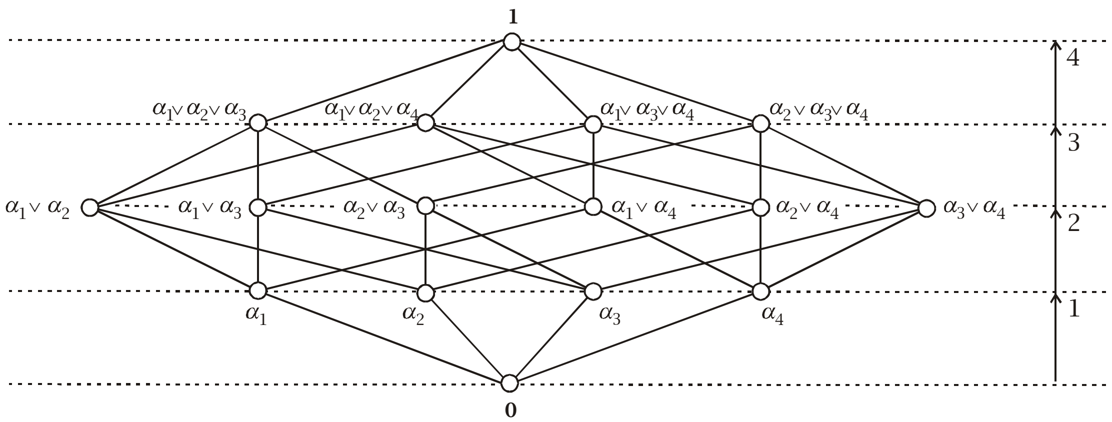
Let us begin with a partition of $\mathbb{I} = [0,1]$ into four mutually disjoint pieces, that we shall call **atoms**: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.



This is a finite partition of the unit interval. A lattice is any partially ordered set (poset), $\mathbb{P}$, in which there are some meets $x \wedge y$ and joins $x \vee y$ of elements $x, y \in \mathbf{P}$. The finite Boolean representation theorem affirms that for any finite Boolean lattice meets correspond to set theoretic intersections, and joins to unions. That is: -

$x \wedge y \Leftrightarrow x \cap y \qquad x \vee y \Leftrightarrow x \cup y$

In this context we will assume the generalised Boolean representation theorem, also known as the Stone representation theorem, which states that the equivalence may be extended to infinite lattices. This equivalence is stated as the principle that every Boolean lattice corresponds to a field of sets.[13] This Boolean lattice that is **derived** from the partition: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, which shall be called its **skeleton**.

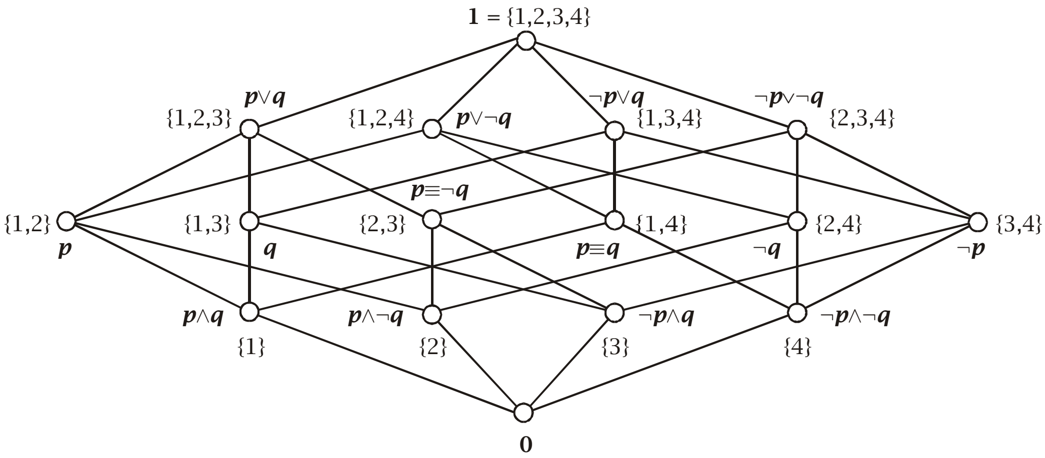


Every such Boolean lattice has a bottom element, **0**, which corresponds to the null set; and a topmost element, **1**, which corresponds to the entire space, here $\mathbb{I} = [0,1]$. The elements **0** and **1** are said to be "distinguished elements" of the lattice. This diagram illustrates that this lattice is a metric space, where the distance between lattice points has measure $d(x,y) = 1$ if there is no intervening lattice point. When two elements $x, y \in \mathbb{P}$ are such that $x > y$ and $d(x,y) = 1$ then we say that *x* **covers** *y*. Atoms are precisely those elements of the lattice $\{\alpha_i\}$ for which meets $\alpha_i \cap \alpha_j, i \neq j$ do not exist as distinct elements of the lattice; that is, $\alpha_i \cap \alpha_j = \varnothing, i \neq j$. Hence, atoms cover the **0** element of the lattice.

In this context the terms **Boolean lattice**, **Boolean algebra** and **Boolean ring** may be treated as synonymous.

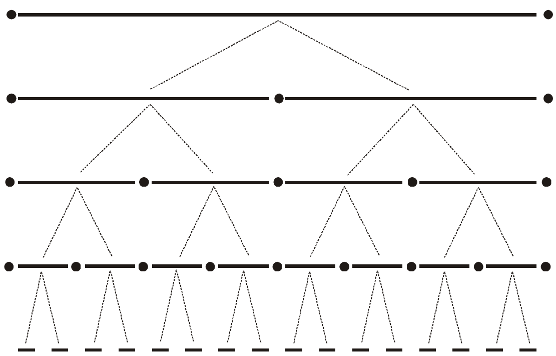
The following diagram demonstrates the isomorphism of the Boolean lattice with the propositional logic of two propositions, *p* and *q*, and with the field of sets generated by the discrete topology on $X = \{1,2,3,4\}$. This is the Boolean algebra, $\mathbf{2}^4 = \{0,1\}^4$.

[13] The Stone representation theorem depends on the Axiom of Choice, but I will subsequently clarify why it is reasonable to assume this on the arithmetical continuum and that it belongs to it as a concept.



The term **atom** is standard in Boolean algebra for mutually disjoint members of a partition that cover the **0** element. An atom is a member of a **topological basis** for the space $\mathbb{I} = [0,1]$ (or any space, *X*, homeomorphic to it). So an atom is not in general a point. The atoms of the continuum are, except in special cases, not points of the continuum.

A Boolean algebra is also a vector space, and the atoms comprise a set of linearly independent vectors that span the space, which is called a **Hamel basis**. Each Boolean algebra is thus related to a tree structure which generates all linear combinations of the vector space as branches; the members of the Hamel base being generated as a subset of these branches: -



0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

The tree form may be denoted $\mathbf{2}^4$ where $\mathbf{2} = \{0,1\}$. We see that $\mathbf{2} = \{0,1\}$ is a factor space of every Boolean algebra; every Boolean algebra is a product of copies of $\mathbf{2} = \{0,1\}$. Since $\mathbf{2} = \{0,1\}$ is a factor of every (finite) Boolean algebra they have the form $\mathbf{B}^{2^n}$, where *n* denotes the number of copies of **2** of which it is a product

Just as there is no real difference between a Boolean algebra, a Boolean lattice and a Boolean ring, because they are different descriptions of the same underlying structure, so the tree form is just another description of this self-same structure.

5.2 Derivation of the lattice from the skeleton

5.2.1 The skeleton

The partition of $\mathbb{I} = [0,1]$ on which the Boolean algebra is constructed is called its **skeleton**. For example, in $\mathbf{B}^4$ the skeleton is its collection of atoms: -

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \Leftrightarrow \{\{1\},\{2\},\{3\},\{4\}\} \Leftrightarrow \{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0)\}$

The relationship between the skeleton and the lattice is a **derivation**. [See also section 18.2] This means that a lattice is uniquely determined by its skeleton and, conversely, we can recover the skeleton from its lattice.[14] This is arguably the most important observation to make about the continuum, as we shall proceed to demonstrate. Although the lattice is a larger structure than the skeleton, all the information contained in the lattice is already contained in its skeleton. The other lattice points correspond to logical and linear combinations of basis elements (atoms), and topologically to other subspaces of the interval $\mathbb{I} = [0,1]$. In the case of the Boolean lattice $\mathbf{2}^n$ the skeleton of is a partition of the unit interval $\mathbb{I} = [0,1]$ into a basis of *n* atoms. This partition is also isomorphic to a structure known as the **Boolean space** of the lattice.

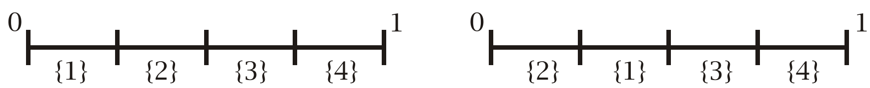
There are several ways in which one can generate a topological basis for a lattice. Another such way is to use **co-atoms**. These are elements of the lattice that are covered by the maximal element, **1**. In our example, $\mathbf{2}^4$, the co-atoms are: -

$\mathbf{1} - \{4\} = \{1,2,3\}$, $\mathbf{1} - \{3\} = \{1,2,4\}$, $\mathbf{1} - \{2\} = \{1,3,4\}$, $\mathbf{1} - \{1\} = \{2,3,4\}$

5.3 Exogenous relationships

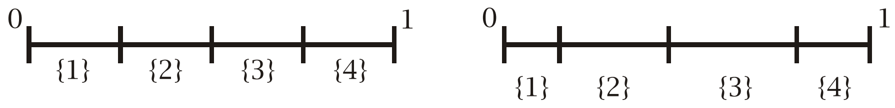
The only relations generated by the derivation of the lattice from the skeleton are those of meets and joins. This has important consequences.

1. If we change the order of the atoms in the skeleton, we derive the same lattice. For example, the following two skeletons of $\mathbb{I} = [0,1]$ have the same Boolean lattice: -



[14] There are lattices which are not Boolean lattices. This result applies to all distributive lattices. Here we are only concerned with Boolean lattices.

2. If we alter the relative size of atoms in the skeleton, we also derive the same lattice: -



The atoms here denote intervals, but here <u>size</u> is an exogenous notion. We cannot derive the measure, length or size of an atom from the mere fact that it is an atom.

The atoms of the skeleton form a collection known as an **antichain**.

5.3.1 Definition, antichain

An ordered set P is an *antichain* if $x \le y$ in P only if $x = y$. (Davey and Priestley [1990] p. 3)

5.4 Chain and antichain

The first observation above demonstrates that order is exogenous to the structure of the lattice. We have labelled the atoms, $\{1\},\{2\},\{3\},\{4\}$; however, these are purely labels, and from the point of view of the lattice, any other labels would do, for example: $a, \varpi, X, \%$. The lattice is derived from the atoms of the skeleton, which is an antichain. To number the members of this antichain and place an order upon them is to convert it into a chain. The lattice generated by a chain of n elements comprises just $n+1$ elements, where successive elements cover each other. So we cannot convert the antichain of the skeleton into a chain; if we do so, we destroy the lattice.

Nonetheless, the arithmetical continuum is required to represent continuous motion, which implies a particle passing through a succession of points or stations as it progresses from start (0) to finish (1) across the interval $[0,1]$. Hence, the skeleton of the arithmetical continuum must simultaneously be <u>both an antichain and a chain</u>. As an antichain the lattice of subspaces is derived from it; as a chain it has the possibility of representing continuous motion.

To accommodate this seemingly paradoxical property, we impose an exogenous relationship of order upon the atoms of the skeleton; that is, we label them as members of a well-ordered ascending chain, yet ignore this information when we derive the lattice from the skeleton of atoms.[15]

[15] It is a claim that this exogenous relation of order is a consequence of the primitive notion of continuity. For continuity the atoms must come in some order by convention, say from left to right. In the notion of a ratio, the bigger part must contain the smaller, so that in a progression from left to right, the smaller is completed first. Hence, this exogenous relation of order is also derived from a primitive notion of continuity.

5.5 Filters and ideals

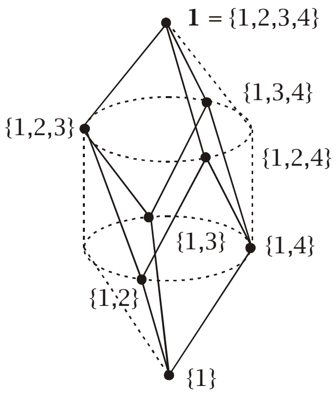
Consider the Boolean lattice $\mathbf{2}^4$ with skeleton: -

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \Leftrightarrow \{\{1\},\{2\},\{3\},\{4\}\} \Leftrightarrow \{(0,0,0,1),(0,0,1,0),(0,1,0,0),(1,0,0,0)\}$$

We may define on this lattice a family of those sets that contain as a subset the atom $\{1\}$. This shall be called its **filter**.

5.5.1 Example

$$\begin{aligned} \text{filter}(1) &= 1 \in x \\ &= \{x \subseteq X : 1 \in x\} \\ &= \{x \subseteq X : \{1\} \subseteq x\} \\ &= \{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,3,4\},\{1,2,3,4\}\} \end{aligned}$$



A filter is also called an ***up-set***, which helps one to visualise its meaning. A filter is said to be a **principal filter** if it has a unique lowest element. In the above example $\{1\}$ is the principal element. This means that every member of the filter contains $\{1\}$ as an element. In finite lattices all filters are principal; however, in infinite lattices this may or may not be the case. The dual notion is an **ideal**, which is a *down-set*.

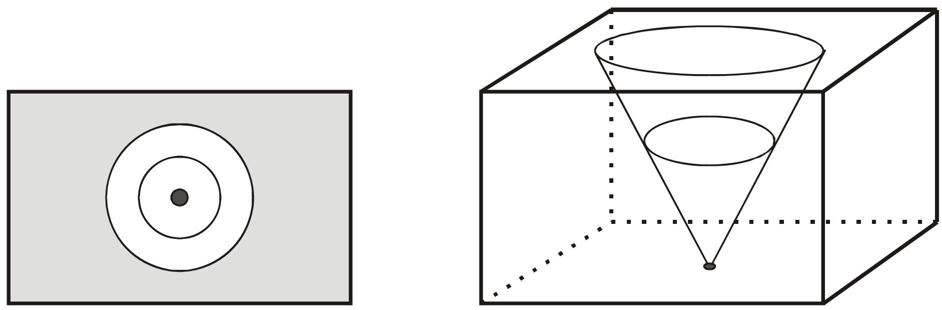
5.5.2 Definition, ideal

An *ideal* is a non-void subset J of a lattice L with the properties

1 $\quad p \in M,\ x \in L,\ x \le p$ imply $p \in M$

2 $\quad p \in M,\ q \in M$ imply $p \vee q \in M$

If we visualise the entire lattice as a "space", a principal filter corresponds to a series of concentric circles radiating from a specific point of this space. A filter may also be pictured as a section through the space.



These diagrams are for heuristic purposes. Every filter contains the entire space, so these diagrams do not show the greatest, ***1****, and least,* ***0****, elements of the lattice.*

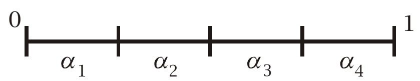
In a finite lattice a filter is called an **ultrafilter** if its principal element is an atom. In an infinite lattice there may not be any principal elements; an ultrafilter in an infinite lattice is one which covers the **0** element. So in infinite lattices ultrafilters take on the role of atoms.

The dual notion is that of a **prime ideal**. An ideal is **prime** if it is only covered by the greatest element, **1**, of the lattice.

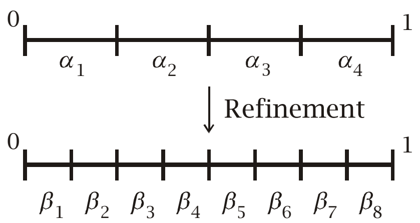
The **Stone space** of a lattice is the collection of all its prime ideals. Let L be a lattice with skeleton P of atoms. Then the Stone space S of prime ideals of L comprises a skeleton of atoms of another lattice L'. The original lattice and the lattice derived from the Stone space are isomorphic: $L \cong L'$. Essentially, they are one and the same structure, though relative to L, the lattice L' is upside down. (L' is an inverted, isomorphic copy of L.). In a finite lattice, the collection of prime ideals of L' is isomorphic to the collection of co-atoms of L.

5.6 Refining the skeleton

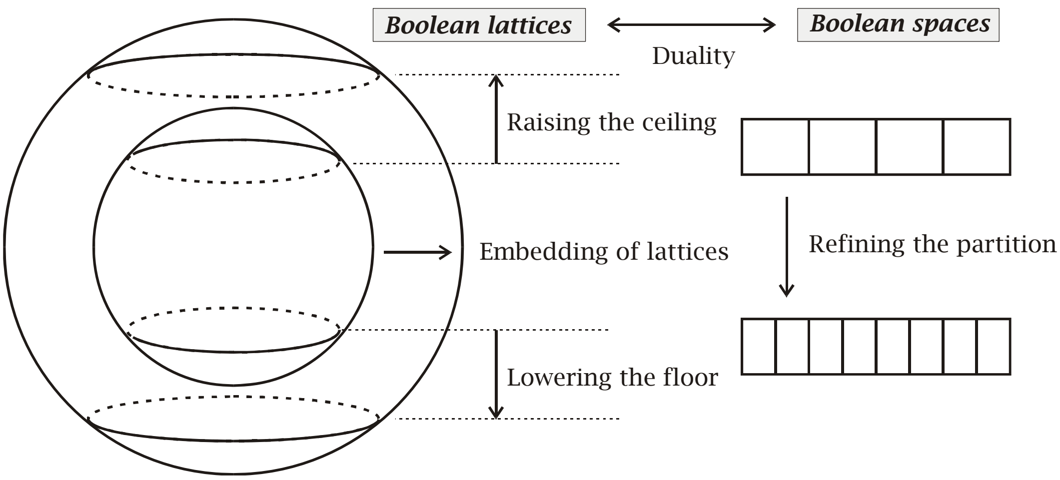
Imagine we have a partition skeleton of the unit interval $\mathbb{I} = [0,1]$ into four basis atoms: -



We may refine the skeleton by splitting the notional atoms; for example, we may split each atom in two, $\alpha_1 = \{\beta_1, \beta_2\}$, and so forth: -



To each skeleton there corresponds a lattice. Visualising these lattices as spheres[16] the process of refining the skeleton leads to an embedding of lattices. I shall also refer to the skeleton of the lattice as its **floor**, and to the collection of prime ideals as its **ceiling**. Thus a refinement of the skeleton is a simultaneous process of **lowering the floor** of the lattice and **raising its ceiling**.



5.4 Boolean-valued models and avoiding semantics

Since we have a correspondence between set theory and Boolean lattices, these Boolean lattices may be described as **Boolean-valued models** of parts of set theory. Finite lattices are only models of finite parts of set theory: not all the axioms of ZFC hold in these models. However, what we are doing here by working directly in lattices that are models of some relevant part of ZFC is circumventing the need to consider the machinery of **semantics** – that is, the relations between the languages of set theory and first-order logic, and their models.

5.7 Finite proof paths and logical consequence

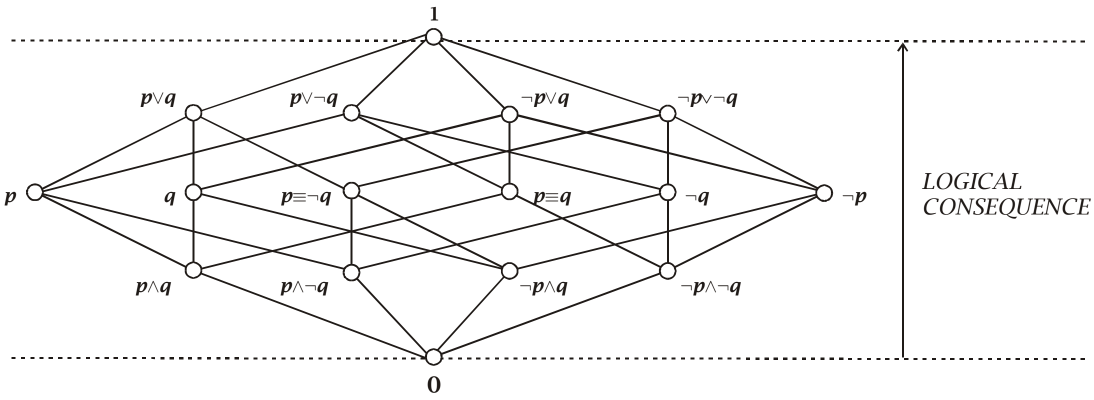
5.7.1 Definition, path

A **path** in a lattice is any chain of connected lattice points, finite or infinite.

Wherever possible I wish in this monograph to avoid the introduction of formal logic. However, some observations on the application of Boolean lattice to logic shall be necessary.

1. In order to turn a Boolean lattice into a model of a particular logic we require lattice points to correspond to propositions and impose an external direction to the lattice. Specifically, we claim that all valid inferences proceed up the lattice. For example: -

[16] This picture is jusfied in the separate notes on infinite lattices.



2. As the diagram illustrates this gives rise to the concept of **logical consequence**. Let *p*, *q* be propositions corresponding to lattice points; then *q* is a consequence of *p* iff there is an upward path in the lattice from p to q (*p* lies below *q*). This means that every filter defines a relation of logical consequence. Every proposition corresponding to a lattice point in $\text{filter}(p)$ is a consequence of the proposition represented by *p*. This also means that all inference in the lattice proceeds by dilution of information: *p* contains more information than any consequence of *p*.
3. Lattices may be finite or infinite. If a lattice is infinite then there are paths in any filter that may be infinite in length. We shall say that *q* is a **logical consequence** of *p* if *q* is any lattice point in $\text{filter}(p)$. We shall say that *q* is a **logical deduction** from *p* if, in addition, the path from the lattice point *p* to *q* is finite (or locally compact)[17]. We denote these relations: $p \vDash q$ for logical consequence, and $p \vdash q$ for logical deduction.
4. A logic is **complete** if $p \vDash q$ iff $p \vdash q$. This means that to every path in any filter is finite (or locally compact). A logic is **incomplete** if $p \vDash q$ but not $p \vdash q$. In such a case there is an infinite path in a filter with no locally compact sub-path.

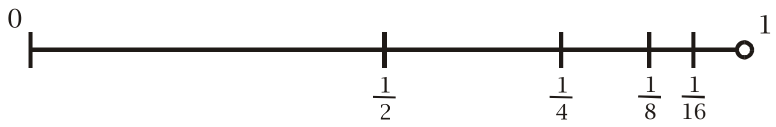
6. The potentially infinite division of the unit interval

6.1 The potentially infinite skeleton

We may observe that no finite skeleton of the unit interval $\mathbb{I} = [0,1]$ can produce a countable dense subset. A finite skeleton is not a dense subset of the unit interval. Therefore, we need to continue the process of refining the partition, or equivalently, lowering the floor of the

[17] Essentially, the path in a derivation is finite. This is because formal proofs are finite structures. The term locally compact is defined below, (Def. 6.1.1). The need to include this is that a path may appear to be infinite, and hence not a derivation, but can be mapped to a finite path. In such cases, the path is locally compact. It is also possible to define logics in which proofs correspond to paths that are of any ordinal length, though this stretches the concept of proof.

lattice, *ad infinitum*. The process of refinement may also be visualised in terms of the paradox of Zeno of the division of the line, to which it is related.



There is a correspondence between this potentially infinite bisection of the unit interval and the number of atoms in the skeleton. After *m* bisections we have a skeleton of 2^m atoms, and a lattice of 2^{2^m} elements.

If we allow this potentially infinite division of the unit interval then we obtain a dense subset, which is isomorphic to $\mathbb{Q}$ in $\mathbb{I} = [0,1]$. The cardinality of the potentially infinite division of the unit interval is $\aleph_0$ since this may be placed in one-one correspondence with the set of all natural numbers, $\mathbb{N}$. So we have a potentially infinite division of the unit interval into $\aleph_0$ pieces.

6.1.1 Definition, locally compact

A topological space is **locally compact** if each of its points has a neighbourhood with a compact closure.

6.1.2 Definition, globally compact

A topological space is **globally compact** if every open cover has a finite subcover.

<u>Remark</u>

Thus by *globally compact* we mean what is often referred to as *compact*. The term *global* is introduced in order to distinguish it from *local compactness*.

6.1.3 Definition, atomic, non-atomic

A lattice is atomic if it has a set of atoms; equivalently, if the maximal element, **1**, is the join of all its atoms. A lattice is non-atomic otherwise.

6.1.4 Results

1. Every globally compact space is locally compact.
2. In a Hausdorff space, every globally compact space is closed and bounded.

Armed with these definitions, we must make two fundamental observations about the potentially infinite division of the continuum: -

1. The resultant partition does not cover the unit interval, since there is always a neighbourhood of 1 omitted. The partition produces a skeleton that is **locally compact** but not **globally compact**. Every proper part of the space is closed and bounded, but taken globally the space is not closed and bounded. The Heine-Borel

theorem does not apply to the space as a whole; the potentially infinite skeleton is not a model of the second order Axiom of Completeness.

2. The correspondent derived lattice is **non-atomic**.

Some further results about non-atomic Boolean lattices will be useful.

6.1.5 Result

Any countably infinite Boolean algebra is non-atomic.

Proof

Let *B* be a countably infinite Boolean algebra, that is of cardinality $\aleph_0$, and suppose *B* is atomic. Let *A* be the set of all atoms of *B* and by the isomorphism theorem $B = \mathbf{P}(A)$. *A* cannot be finite, for then *B* would be finite. Therefore, *A* is countably infinite. Then $\mathbf{P}(A)$ is an atomic Boolean algebra of cardinality $2^{\aleph_0} > \aleph_0$. This contradicts the assumption that *B* is of cardinality $\aleph_0$. Hence, *B* cannot be atomic.

The atomless lattice derived from the potentially infinite skeleton of $\aleph_0$ parts corresponds to to the atomless algebra, *S*, of (propositional) statement bundles. In fact, this atomless structure is strictly not a Boolean algebra. Being atomless it lacks a greatest element, **1**, which in a Boolean lattice is the join of all the atoms. But this is the only way in which it differs from a Boolean lattice; the structure is well defined and is called a **generalised Boolean algebra**.

6.1.6 Result

Every atomless Boolean algebra with more than one element must be infinite.

Proof

The unit 1 is different from zero, so there is a non-zero element p_1 strictly below 1; otherwise, 1 would be an atom. Because p_1 is not zero, there must be a non-zero element strictly below p_1; otherwise, p_1 would be an atom. Continue in this fashion to produce an infinite, strictly decreasing sequence of elements $1 > p_1 > p_2 > \ldots$.

6.1.7 Result

Any two countably infinite Boolean algebras (without atoms) are isomorphic.

Proof

It can be shown that the order of any finite Boolean algebras is 2^n for some $n \in \mathbb{N}$, and any two finite Boolean algebras with the same number of elements are isomorphic.

Let *A*, *B* be two countably infinite Boolean algebras. Let $A = \{a_0, a_1, a_2, ...\}$ and $B = \{a_0, a_1, a_2, ...\}$ be enumerations of *A* and *B* respectively. The proof will be a "back and forth" argument.

The proof proceeds inductively on the order of subalgebras.

For $n = 0$ let $A_0 = \{0,1\}$ and $B_0 = \{0^*, 1^*\}$; then $A_0 \cong B_0$.

For the induction step, suppose that the result is true for all $k < n$; that is $A_k \cong B_k$.

Suppose *n* is even.

Let $a_j \in A - A_{n-1}$ be the element with smallest index *j*, and let A_n be the subalgebra generated by $\{a_j\} \cup A_{n-1}$.

Then there is an element $b_m \in B - B_{n-1}$ such that the subalgebra B_n generated by $\{b_m\} \cup B_{n-1}$ is isomorphic to A_n.

Suppose *n* is odd.

In this case select first $b_m \in B - B_{n-1}$ to generate B_n, and the claim is that there is an element $a_j \in A - A_{n-1}$ that generates $A_n \cong B_n$.

So the induction step holds and so $A = \bigcup A_n \cong \bigcup B_n = B$.

Remark

We should prove that both algebras pair off atoms in the respective finite subalgebras. (This proof is based on Komjáth and Totik [2000]. There is also a proof in Givant and Halmos [2002] p.135)

The potentially infinite partition of the unit interval, its skeleton, has $\mathbb{N}$ non-atomic pieces. The derived lattice shall be denoted $\mathbf{2}^{\mathbb{N}}$. In the literature, the distinction the potential and actual infinite is implicit only. Authors do not systematically distinguish between $\mathbb{N}$ and ω. In contexts where the distinction is essential, we see in place of $\mathbb{N}$ the symbol $<\omega$. Thus, in the literature the non-atomic countably infinite generalised Boolean algebra is denoted $\mathbf{2}^{<\omega}$. The symbol $\mathbf{2}^{(\omega)}$ is also used.

6.1.8 Result, The potential and actual infinite (+)[18]

Analysis distinguishes between the potential and actual infinite.

Proof

The structure $2^{<\omega}$ is locally compact and being open is not globally compact. Its skeleton is a locally compact potential infinity. The Cantor set $\mathbf{2}^{\omega}$ is globally compact and actually infinite.

[18] The mark (+) indicates a result in this paper that to the best of my knowledge has not been stated in any other paper.

6.2 The character of the atomless countably infinite generalised Boolean algebra

The non-atomic countably infinite generalised Boolean algebra, $\mathbf{2}^{\mathbb{N}}$, has "paradoxical" properties. As it is non-atomic, there is no simple way to picture the partition of the unit interval on which it is based. Unlike the finite case, we cannot draw the lattice. To do so would require a complete set of atoms, the very thing we do not have. However, let us suppose that we have a set of what we shall call **notional atoms** comprising some floor of the lattice, $\mathbf{2}^{\mathbb{N}}$. Because the lattice is non-atomic, it is always possible to lower the floor of the lattice; this corresponds to a refinement of the partition. Nonetheless, when we do so the new lattice is a copy of the old one. The lattice $\mathbf{2}^{\mathbb{N}}$ may be embedded into another lattice that is a copy of itself. I call this the **flip-flop property** of the lattice, $\mathbf{2}^{\mathbb{N}}$. It is a weird property, but not a formal paradox. This property arises from the fact that no partition of the unit interval into $\mathbb{N}$ pieces is actually possible; here $\mathbb{N}$ represents a potential infinite, that is, the possibility of continuing the process of partition indefinitely.

7. The actually infinite skeleton of the continuum, and the one-point compactification

7.1 The one-point Alexandroff compactification

It is at this point when we must consider how to divide the interval $[0,1]$ into an actually infinite number of partitions equinumerous to ω that the tangible distinction between the potentially infinite $\mathbb{N}$ and the actually infinite ω makes a difference of momentous significance. The collection $\mathbb{N}$ is literally incapable of dividing the interval $[0,1]$ into an infinite number of segments for the reason that it represents a potential infinity. If we start numbering the partitions we will never have done, because $\mathbb{N}$ is unbounded above. $\mathbb{N}$ is a locally compact [Definition 6.1.1] but not globally compact [Definition 6.1.2] set. $[0,1]$ is bounded. A partition of $[0,1]$ by $\mathbb{N}$ segments is impossible because we are trying to divide the unbounded into the bounded.

The partition of $[0,1]$ requires a 1-point **Alexandroff compactification** [Definition 4.1.2]. When the space is Hausdorff, as $\mathbb{R}$ is, the compactification is called the **Stone-Cech compactification**.

7.1.1 The Stone-Cech compactification

A compactifiction Y of a locally compact Hausdorff space X is a **Stone-Cech compactification** of X if every continuous mapping from X into a compact Hausdorff space Z can be extended to a continuous mapping from Y into Z. (From Givant, Steven and Halmos, Paul [2009] p. 413.)

Since the Hausdorff condition follows from the Commpleteness Axiom, the Stone-Cech compactification is a consequence of this axiom – it is another expression of the Heine-Borel theorem. The half-open interval $[0,1)$ is locally compact but not closed or bounded above. In order to close it, we need to adjoin to it the neighbourhood of the point 1, here represented by $\{\infty\}$. That is, we may write, $[0,1]=[0,1)\cup\{\infty\}$.

The interval $[0,1]$ is the sub-manifold of $\mathbb{R}$ upon which we are currently attempting to define a scaffold (skeleton) or partition of actually ω parts. A subdivision of $[0,1)$ would correspond nicely to a partition by $\mathbb{N}$ parts precisely because the sub-manifold is open and unbounded just as $\mathbb{N}$ is unbounded. But $\mathbb{N}$ is insufficient to partition $[0,1]$.

7.1.2 Result

Every discrete space is locally compact, but not compact if infinite. (Bourbaki [1989a] p. 90)

<u>Corollary</u>

$\mathbb{N}$ is locally compact but not compact.

The standard partition may be found in any appropriate text. I take it from Davey and Priestley [1990] (p.197): -

7.1.3 One-point compactification of a countably discrete space

Let $\mathbb{N}_\infty = \mathbb{N}\cup\{\infty\}$.

Let $U \subseteq \mathbb{N}_\infty$.

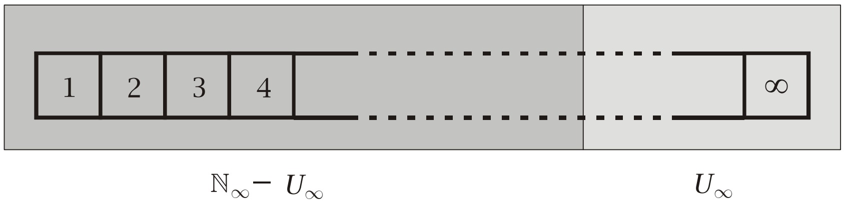
Let T be the topology on $\mathbb{N}_\infty$ given by

$$U \in T \text{ if } \begin{cases} \infty \notin U \\ \infty \in U \text{ and } \mathbb{N}_\infty - U \text{ is finite} \end{cases}$$

This can be shown to be a topology. (See Givant and Halmos [2009]). A subset $V \subseteq \mathbb{N}_\infty$ is clopen (both closed and open) iff V and $\mathbb{N}_\infty - V$ are in T. The clopen sets of $\mathbb{N}_\infty$ are the finite sets not containing ∞ and their complements. It can be shown that $\mathbb{N}_\infty$ is totally disconnected.[19]

[19] Givant and Halmos write: "A less trivial collection of examples consists of the one-point compactifications of infinite discrete spaces. Explicitly, suppose a set X with a distinguished point x_0 is topologized as follows: a subset of X that does not contain the point $\{x_0\}$ is always open, and a subset that contains x_0 is open if and only if it is cofinite. It is easy to verify that the space X so defined is Boolean. For instance, a subset of X is clopen if and only if it is either a finite subset (of X) that does not contain $\{x_0\}$ or else a cofinite subset that contains x_0; indeed, a subset and its complement are both open just in case one of them (the one that contains x_0) is cofinite. The clopen sets form a base for the

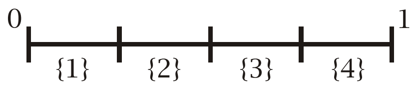
This definition results in a compactification of $\mathbb{N}$. This derives from the definition of the topology on $\mathbb{N}_\infty$.



The topology is defined in such a way that, if an open set, U_∞, covers $\{\infty\}$ then what remains, $\mathbb{N} - U_\infty$, must be finite; hence every open cover has a finite subcover. This mirrors the construction of the finite cover in the Heine-Borel theorem.

7.2 Being precise about the skeleton

There is a subtle and very important point to make here. Consider the skeleton of the unit interval comprising just four atoms: -

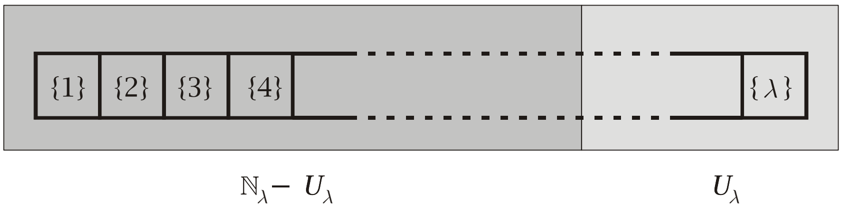


By analogy in our actually infinite partition of the unit interval, the atoms should not be labelled, 1, 2, 3, … ∞, but rather, $\{1\},\{2\},\{3\},\ldots,\{\infty\}$. As a model of the continuum, in the expression $\{1\}$, it is the whole set $\{1\}$ which represents the atom of the partition; the symbol 1 denotes the content of that atom; this content is the individual of which the set is its collection.

So $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ is not the skeleton of the closed unit interval. We will designate the correct skeleton by $\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\},\ldots\} \cup \{\{\lambda\}\} = \{\{1\},\{2\},\{3\},\ldots,\{\lambda\}\}$ where the "point at infinity" representing the neighbourhood of 1 in the unit interval is designated $\{\lambda\}$. The members of this skeleton, $\{1\},\{2\},\{3\},\ldots,\{\lambda\}$, comprise an unordered set, and the symbol $\{1\}$

---

topology because every open set that contains x_0 is clopen, while every open set that does not cotnain x_0 is the union of its finite subsets." (Givant and Halmos [2009] p. 301, where the discussion continues.) We also have Alexandroff's Theorem: Let *X* be a locally compact space. (1) Then there exists a compact space X_∞ and a homeomorphism *f* of *X* onto the complement of a point $x_0 \in X_\infty$. (2) If X_∞' is another compact space such that there is a homeomorphism f_1 of *X* onto the complement of a point in X_∞', then there is a unique homeomorphism *g* of X_∞ onto X_∞' such that $f_1 = g \circ f$. [Source is Bourbaki [1989a] p. 92] is merely a label for an atom of this skeleton. It is a label for a portion of space. This set has to be unordered because we wish to construct a Boolean algebra over it. But since we wish to use this skeleton as a model of the arithmetical continuum, we must also impose an external relationship of order as well. So we regard $\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\},\ \dots\ ,\{\lambda\}\}$ as internally unordered but externally ordered and hence externally isomorphic to $\mathbb{N}_\infty = \{1,2,3,\ \dots\ ,\infty\}$. The topology of the skeleton is illustrated by this diagram: -



Since the unit interval, $\mathbb{I} = [0,1]$, is the complete algebra derived from its skeleton, which here forms its countable dense subset, the compactification of the unit interval follows from the 1-point compactification of its skeleton, $\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\},\ \dots\ ,\{\lambda\}\}$.

7.2.1 Definition, derived set (+)

We denote the complete Boolean algebra by $\mathbf{2}^{\mathbb{N}_\lambda}$ that is derived from the one-point compactification of the continuum, $\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\},\ \dots\ ,\{\lambda\}\}$, **the Derived set**.

7.2.2 Theorem (+)

The Derived set is homeomorphic to the continuum, $\mathbb{R}$.
(See section 18.2 for a discussion and demonstration of this result.)

Thus we have a distinction between the Cantor set, $\mathbf{2}^\omega$, and the Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$. The Cantor set is a representing set for the continuum because it has continuum many points and shares many common properties with it. However, it is not a continuous set and so cannot be homeomorphic to the continuum. The Cantor set represents a family of models that can be distinguished by combinatorial set theoretic principles. Among these combinatorial structures there is just one that is entailed by the Axiom of Completeness. This is the combinatorial structure determined by the Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$.

Although $\mathbb{N}_\lambda$ and ω are equinumerous, both having cardinality $\aleph_0$, it is a mistake to conclude that they represent the same structure in regard to the partition of the interval $[0,1]$ into actually ω parts. The derived set of ω is the Cantor set, $\mathbf{2}^\omega$. As a skeleton of the arithmetical continuum, a partition of ω parts is wholly ambiguous, which is why the question, "How many points are there in its derived set?" is unanswerable. To resolve the continuum hypothesis we need a definite model of the continuum that is one that is homeomorphic to it. We must also keep in mind the distinction between a chain and an antichain. Not only does the structure $\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\}, \dots ,\{\lambda\}\}$ represent an actually infinite partition of the unit interval, as opposed to a potentially infinite one, it also represents that partition as having an exogenous relation of order, so that we take its atoms, $\{1\},\{2\},\{3\}, \dots ,\{\lambda\}$ in succession as numbering individual segments of the unit interval taken in succession. To recap, we must actually "forget" this additional structure when we generate the lattice because otherwise when we derive the lattice from the skeleton we will obtain just another chain.

Hence, we need yet a third notion of the infinite, to denote a potentially infinite collection of unordered atoms – a potentially infinite antichain [Definition, 5.3.1]. I shall denote this collection by μ; μ is collection of $\aleph_0$ elements without any order relation upon them.

7.2.3 Summary, three distinct notions of the countably infinite (+)

ω	Actually infinite chain of ordinals.
$\mathbb{N}$	Potentially infinite chain of natural numbers.
μ	Potentially infinite antichain of unordered elements.

The status of μ and $\mathbb{N}$ as sets is disputable. According to the popular theory that every mathematical entity is a set, they must either be sets or just not exist. Levy [2002] does not include $\mathbb{N}$ in his text at all, which suggests that he denies that it exists as a separate entity from ω. However, we can see that the one-point compactification of the discrete space $\mathbb{N}$ makes no sense at all if $\mathbb{N}$ does not exist and the existence of $\mathbb{N}$ <u>as distinct from</u> $\mathbb{N}_\infty$ is implicit in the entire theory of Boolean lattices. As intensions $\omega, \mathbb{N}_\infty$ and $\mathbb{N}_\lambda$, we have distinct concepts: -

7.2.4 Distinct variants of the skeleton of the unit interval

1. ω is the least limit ordinal – an actually infinite collection of all finite ordinals: $\omega = \{0,1,2,3,\dots\}$. There is no ∞ in this set.[20] This shall be called the **ambiguous skeleton** of the real line.
2. $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\} = \{0,1,2, \dots , \infty\}$ is the one-point compactification of the discrete space $\mathbb{N}$ and a model of the actually infinite partition of the interval $[0,1]$ in which the symbols 0, 1, 2, … ∞ are arbitrary labels for the atoms of the skeleton. We call this the **skeleton of the discrete space**.

[20] The distinctions between $\mathbb{N}, \mathbb{N}_\infty$ and ω raise the question of non-standard models of arithmetic in which we see additional elements tagged onto the set $\mathbb{N}$ and making the resultant model non-categorical for $\mathbb{N}$. This relates to ω-consistency. For a description of non-standard models of arithmetic, see Boolos and Jeffrey [1980] Chapter 17.

3. $\mathbb{N}_\lambda = \mu \cup \{\{\lambda\}\} = \{\{1\},\{2\}, \ldots, \{\lambda\}\}$ is the partition of the interval $[0,1]$ in which $\mu = \{\{1\},\{2\},\{3\}, \ldots\}$ represents a potentially infinite antichain. The skeleton, $\mu_\lambda = \mu \cup \{\lambda\}$ is intrinsically unordered but, as a model of the skeleton of the continuum, it must also be well-ordered extrinsically by placing it into one-one correspondence with elements of $\mathbb{N}$. This shall be called the **canonical skeleton of the continuum**.

This model of the arithmetical continuum assumes the Axiom of Choice. This is because $\mathbb{N}_\lambda = \mu \cup \{\{\lambda\}\} = \{\{1\},\{2\}, \ldots, \{\lambda\}\}$ which acts as the skeleton for the arithmetical continuum must simultaneously be externally well ordered. This external well ordering is supplied by the Axiom of Choice.

7.2.5 The Axiom of Choice

Every set can be well-ordered.[21]

We observe that the canonical skeleton of the continuum contains at least one urelement and is a non-standard model of arithmetic. Whatever the labels $\{1\}, \{2\}, \{3\}, \ldots$ represent, and for the present we may think of them as denoting intervals, the label $\{\lambda\}$, standing for the infinitesimal neighbourhood of 1 in the unit interval, is something quite different.

7.3 The relationship of the Axiom of Choice to the Axiom of Completeness

In textbooks of set theory there is no formal treatment of the Axiom of Completeness. For example, Levy [2002] does not mention it. However, the Completeness Axiom is implicit everywhere and an equivalent to it is needed whenever a recursive potentially infinite process needs to be completed, just on analogy with the Dedekind cut. Whenever this is required it is always the Axiom of Choice that is invoked within the context of ZF theory as a whole, which already has the Axiom of Infinity. But the Axiom of Infinity is not enough to supply completeness arguments. Consider the following remark from Givant and Halmos [2009]: -

> There is a close connection between complete ideals and the "cuts" that play a crucial role in Dedekind's classical construction of the real numbers from the rational numbers. (Givant and Halmos [2009] p.206)

[21] This is actually the Well-ordering Principle which is equivalent to the Axiom of Choice. A set that is well-ordered has a first element. Since every finite set can be well-ordered the Well-ordering Principle is needed for infinite sets. It is not intuitively obvious that any infinite set can be ordered so as to have a first element.

The complete ideals that they discuss can only be established on the basis of the Axiom of Choice.

7.3.1 Result, The one-point compactification requires the Axiom of Choice (+)

The definition of the topology T on $\mathbb{N}_\infty$ [7.1.3. above] makes an implicit use of the Axiom of Choice.

Proof

The construction: -

$\infty \in U$ and $\mathbb{N}_\infty - U$ is finite $\Rightarrow$ U is open in T

requires that we are able to pick out the element ∞ from $\mathbb{N}_\infty = \{0,1,2,3, \ldots, \infty\}$ which since it is an actually infinite set is not possible unless we have a choice function, or equivalently, unless $\mathbb{N}_\infty$ is an well-ordered set. The collection $\mu_\infty = \mu \cup \{\infty\}$ is an unordered anti-chain. It requires the Axiom of Choice to give it an alternative description as an ordered chain.

On the continuum the Axiom of Choice therefore derives its validity from the second-order Axiom of Completeness and is a consequence of it. It is part of the necessary first-order set theoretic gadgetry required to construct a model of the arithmetical continuum.

By the same token, however, the model $\mathbb{N}_\infty = \{0,1,2,3, \ldots, \infty\}$ is a non-standard model of arithmetic. It is not categorical for arithmetic. The categorical model of arithmetic requires just $\mathbb{N} = \{0,1,2,3, \ldots\}$ from which all elements that are not numbers are excluded. As a model of the continuum $\mathbb{N}_\lambda = \mu \cup \{\{\lambda\}\} = \{\{1\},\{2\}, \ldots, \{\lambda\}\}$ does not contain any numbers; it contains an infinite collection of a base partition of atoms of the unit interval which forms a dense subset in the arithmetical continuum. It is an exogenous fact that this base partition may be placed in one-one correspondence with the set $\mathbb{N}_\infty = \{0,1,2,3, \ldots, \infty\}$. That $\mathbb{N}_\lambda = \{\{1\},\{2\}, \ldots, \{\lambda\}\}$ must be exogenously well ordered follows from the fact it is a continuum, and the correspondent first-order set theoretic interpretation of the arithmetical continuum must also have a tool for this. This tool is the Axiom of Choice.

7.3 Representation of the point at infinity

Since there is an exogenous relation of order on the skeleton $\mathbb{N}_\lambda = \{\{1\},\{2\}, \ldots, \{\lambda\}\}$ it is usual to represent the point at infinity $\{\{\lambda\}\}$ as a countably well-ordered set that well orders the antichain, $\mu = \{\{1\},\{2\},\{3\}, \ldots\}$. Hence we may also write the point at infinity as $\{\{\lambda\}\} = \{\{\mathbb{N}\}\}$, the 1-point compactification of the discrete space as $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \{1,2,3, \ldots, \mathbb{N}\}$ and the one-point compactification of the skeleton of the real line as $\mathbb{N}_\lambda = \mu \cup \{\{\mathbb{N}\}\}$. This displays these structures as well ordered sets. We will subsequently display a very close relation between the atom $\{\{\lambda\}\}$ and Mahler's number: $M = 0.12345678910111213141516\ldots$ [Section 16.4.1]

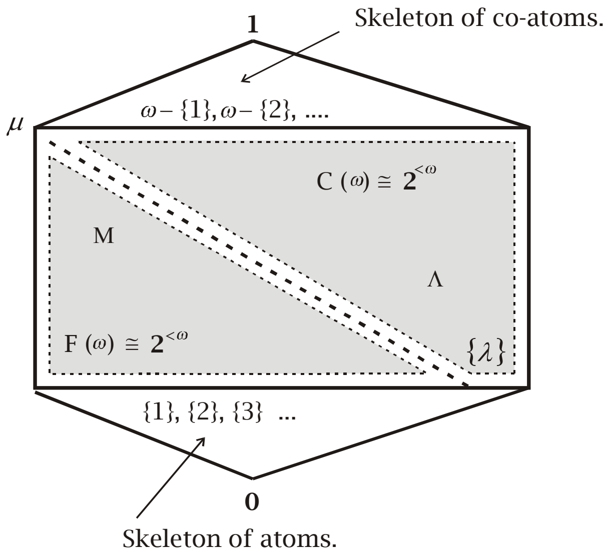
7.4 Implications for set theory as a model of number theory

By the arguments above the Axiom of Choice is a valid principle in the context of the set theory of the real line, but it does not follow that it is valid in the context of number theory, since the arithmetical continuum is not a categorical structure for number theory. It may be observed that in set theory with choice (ZFC) the Axiom of Choice ensures that every set is well ordered. In the absence of the Axiom of Choice the principle of transfinite induction does not apply to <u>all sets</u>: the axiom of infinity alone is not sufficient to extend the principle of induction to all sets. In number theory there is a single principle of complete induction that categorically applies to all natural numbers; on the arithmetic continuum there is an analog principle of transfinite induction that is entailed by the Axiom of Choice. Therefore, the Axiom of Choice is closely related to the principle of complete transfinite induction on actual infinite collections and it derives its validity from the model of the arithmetical continuum. Since it is true that in the absence of the Axiom of Choice, the principle of transfinite induction does <u>not</u> apply to all sets, it is not correct to regard the principle of transfinite induction as <u>stronger</u> in its deductive consequences to the principle of complete induction; the two principles are not comparable. Number theory is equipped with a principle of complete induction defined on the entire potentially infinite collection of natural numbers; the arithmetical continuum is equipped with a principle of complete transfinite induction defined on the entire collection of actually infinite sets, but this principle not only requires the regular axioms of ZF, but also specifically the Axiom of Choice.

The well ordering of the set of real numbers, $\mathbb{R}$, may be derived from the well-ordering of the skeleton $\mathbb{N}_\lambda = \{\{1\},\{2\},\ldots,\{\lambda\}\}$. Subject to the Axiom of Completeness, the set of all real numbers may be well-ordered.

8. The derived set of the one-point compactification

We will use the following diagrammatic representation of the Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, as a model of the arithmetical continuum, and it will be important to explain its features in detail.



The structure of this derived set of the actually infinite skeleton $\mathbb{N}_\lambda$ has many fascinating properties that require deep investigation. For the present we list only those that we can infer from the information we have to hand.

8.1 The distinction between the Derived set and the Cantor set

The whole diagram may be taken as a representation of the entire Derived set, but it is once again very important to emphasise from the outset that the principle "identical up to isomorphism" does not apply in this case. The Cantor set, $\mathbf{2}^\omega$, as a model of the continuum is undetermined. The skeleton $\mathbb{N}_\lambda = \mu \cup \{\lambda\}$ has as many members as ω, but it is not an equivalent order type to it.

$\omega = \{0,1,2,3, \ldots\}$

$\mathbb{N}_\lambda = \mu \cup \{\{\lambda\}\} = \{\{1\},\{2\}, \ldots , \{\lambda\}\}$

The set of all ordinals, ω, <u>has no last member</u>, whereas the actually infinite skeleton, $\mathbb{N}_\lambda$ has a last member, $\{\lambda\}$, though even this fact is <u>not internal</u> to the structure, but <u>external</u> to it, being forced upon us by the need to look on $\{\lambda\}$ as a label for the neighbourhood of 1 in the unit interval, $[0,1]$, thus placing it <u>after</u> all the other labels $\{1\},\{2\},\{3\}, \ldots$ which correspond to intervals within the disjoint neighbourhood of 0 in $[0,1]$, which we can denote $[0,1)$. $\mathbb{N}_\lambda$ is order-isomorphic to $\omega+1$.

The distinction that we draw here, between the Cantor set, $\mathbf{2}^\omega$, and the derived set of the skeleton, $\mathbf{P}(\mathbb{N}_\lambda)=\mathbf{2}^{\mathbb{N}_\lambda}$, <u>must be correct</u>. At the least we know that the Cantor set <u>is not homeomorphic</u> to the arithmetical continuum, since it is nowhere dense and totally disconnected. We will prove that Derived set, $\mathbf{P}(\mathbb{N}_\lambda)=\mathbf{2}^{\mathbb{N}_\lambda}$, is homeomorphic to the closed unit interval, $[0,1]$. (Section 18.2) It is because the Cantor set is not homeomorphic to the continuum that the cardinality issue has hitherto remained unresolved.

8.2 The skeleton is not homogeneous

The skeleton, $\mathbb{N}_\lambda$, contains a potentially infinite proper subset, $\mu=\{\{1\},\{2\},\{3\},\ldots\}$. As members of the actually infinite skeleton, $\mathbb{N}_\lambda$, the labels $\{1\},\{2\},\{3\},\ldots$ together with $\{\lambda\}$, represent <u>atoms</u> of the Derived set, $\mathbf{P}(\mathbb{N}_\lambda)=\mathbf{2}^{\mathbb{N}_\lambda}$. Since there are $\aleph_0$ of these the $\{1\},\{2\},\{3\},\ldots$ are <u>not points</u> of the continuum, but collectively comprise a dense subset of the continuum, isomorphic to $\mathbb{Q}$, forming a basis for the arithmetical numbers [Section 9]. However, $\{\lambda\}$ does not belong to this basis, but is an additional atom representing the neighbourhood of 1. Therefore, $\{\lambda\}$ and $\{1\},\{2\},\{3\},\ldots$ are not homogeneous. The collection $\{1\},\{2\},\{3\},\ldots$ may be placed externally in one-one correspondence with the natural numbers: -

$\{1\}\leftrightarrow 1\quad \{2\}\leftrightarrow 2\quad \{3\}\leftrightarrow 3\quad \ldots$

But $\{\lambda\}$ cannot be added to this list. The natural numbers have no last member, so $\{\lambda\}$, as the last member of $\mathbb{N}_\lambda$, cannot be included. This makes $\{\lambda\}$ into an urelement. $\mathbb{N}_\lambda$ is a non-standard model of arithmetic.

$\{\lambda\}\notin\mathbb{Q}$; $\{\lambda\}$ it is not an element of the dense subset of the continuum. This may be proved: if it denotes member of the dense subset then it becomes identical to one of the other members, so we can write $\{\lambda\}=\{q\}$ for some $q\in\mathbb{Q}$. This is a contradiction, because this is precisely the possibility that the actual infinite skeleton excludes.

In the one-point compactification, we adjoin $\{\lambda\}$ to the potentially infinite skeleton $\mu=\{\{1\},\{2\},\{3\},\ldots\}\cong\mathbb{N}$ in order to complete it.

8.3 The ideal of all finite subsets

Within the derived set we have an ideal of all finite subsets of $\mu=\{\{1\},\{2\},\{3\},\ldots\}\cong\mathbb{N}$. This is denoted $\mathbf{Fin}=\mathrm{F}(\omega)$ or $\mathrm{F}(\mathbb{N})$, where F is an operation of taking all finite subsets; **Fin** is standard in the literature, as is the representation $\mathbf{2}^{<\omega}$ to which it is isomorphic. Evidently $\mathrm{F}(\omega)=\mathrm{F}(\mathbb{N})=\mathrm{F}(\mu)$ because the finite subsets of $\omega,\mathbb{N}$ and μ are the same. This also permits the identification $\mathbf{Fin} = \mathbf{2}^{<\omega}$ and **Fin** as the potentially infinite proper sub-tree of the actually infinite Cantor set, $\mathbf{2}^{\omega}$.

Fin is a family of ideals, a poset based on a supraset relation. An example of one chain in this family is the following: -

$\mathbf{P}(\{1\}) \subset \mathbf{P}(\{1,2\}) \subset \mathbf{P}(\{1,2,3\}) \subset$

It is a poset because we also have other chains in the family, for example: -

$\mathbf{P}(\{1\}) \subset \mathbf{P}(\{1,3\}) \subset \mathbf{P}(\{1,3,4\}) \subset$

This makes it clear that **Fin** has no maximal element $m \in \mathbf{Fin}$. So **Fin** is incomplete and there is no join in **Fin** representing the union of all finite subsets of $\mathbb{N}$. Nonetheless, the Bolean representation theorem entails that there is a maximal ideal, which we will denote by M. In fact, within the derived set, $\mathbf{P}(\mathbb{N}_\lambda) = \mathbf{2}^{\mathbb{N}_\lambda}$, M is not only a prime ideal but also a principal one. This is because $\mu = \{\{1\},\{2\},\{3\}, \ldots\}$ is the complement in $\mathbf{2}^{\mathbb{N}_\lambda}$ of $\{\lambda\}$: $\mu = \{\lambda\}'$. Hence, since $\{\lambda\}$ is an atom, μ is a co-atom or, equivalently, principal element of the prime ideal of all finite subsets of the skeleton. Thus we have $\mathrm{M} = \mathbf{P}(\mu)$. The relationship of **Fin** to M is as $\mathbb{N}$ to ω. **Fin** is the **interior** of M. **Fin** is open, M is closed and bounded. M is the closure of **Fin**.

8.4 The filter of all cofinite subsets

There is also a filter, $\mathbf{Cofin} = \mathrm{C}(\omega)$, within the derived set comprising of all infinite subsets of $\mathbb{N}_\lambda \sim \aleph_0$ obtained by set difference of ω and members of $\mu = \{\{1\},\{2\},\{3\}, \ldots\}$. These may be represented as co-finite subsets of ω and may be enumerated as $\omega - \{1\}, \omega - \{2\}, \omega - \{3\}, \ldots$ $\{\lambda\}$ is an atom of the skeleton, hence it has a filter: $\mathrm{filter}(\lambda)$. This filter contains every cofinite subset of the derived set, so we may write, $\mathbf{Cofin} = \mathrm{C}(\omega) = \mathrm{filter}(\lambda)$. This also displays this filter as a principal filter within the Derived set.

8.5 A principle of complementarity

We now encounter a seemingly paradoxical property of the Derived set – to do with the way in which the potential and actual infinite combine. Since **Fin** is unbounded in itself, it is non-atomic. Its cardinality is $\aleph_0$ and we have already demonstrated that there can be no atomic Boolean algebra of this cardinality. This forces us to look on the members of the skeleton of the arithmetical continuum in two different ways, which we may call a **principle of complementarity**. This is to do with how we interpret the members of $\mu = \{\{1\},\{2\},\{3\}, \ldots\}$.

8.5.1 The actual and atomic

From the point of view of the derived set as a whole (the continuum), the members of $\mu = \{\{1\},\{2\},\{3\},\ldots\}$ together with $\{\lambda\}$ are atoms. This means that they cover the **0** of the lattice, and between any given member $\{n\}$ of the skeleton and **0** there are no other elements of the lattice. This makes M into a Boolean algebra in its own right, with maximal element, $\mathbf{1} = \mu$. But for that to be the case we must regard $\mu = \{\{1\},\{2\},\{3\},\ldots\}$ as a completed totality, an actually infinite collection. So this has a claim to externally identify $\mu = \{\{1\},\{2\},\{3\},\ldots\}$ with ω. It is an actually infinite collection that has no infinite subsets. In addition to μ we have from this viewpoint one other atom, $\{\lambda\}$, so this is a skeleton of order type $\omega+1$ and of cardinality $\aleph_0$.

8.5.2 The potential and non-atomic

From the point of view of the potentially infinite part of the skeleton, from within $\mu = \{\{1\},\{2\},\{3\},\ldots\}$, the members are not atoms, because there are no atoms in this structure. They represent notional atoms, a "snapshot" of the structure, or a part of that structure, and implying that every meet of every combination of these notional atoms is allowed. So the floor of the lattice may be lowered. We may also call this floor a **pseudo basis** for the topology of the continuum. It is a pseudo basis because it is not complete and any two putative members of the basis have a non-empty intersection. The function of this potentially infinite skeleton is to provide a countably dense subset of the continuum, so from this perspective this is a set of order type, η. It is countable, dense and has no end-points. What this also means is that from within **Fin** we cannot visualise the set $\mu = \{\{1\},\{2\},\{3\},\ldots\}$ as linearly ordered starting with the "first" notional atom next to 0 in the interval $[0,1]$. The skeleton embraces the idea of a procedure for systematically generating the notional atoms, but this is a process that cannot be completed, so that between $\{1\}$ and 0 in the interval it is always possible to introduce a more refined interval.

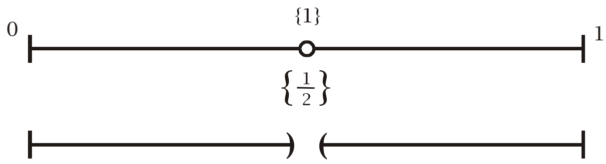
8.6 Quotient structures

Corresponding to the two perspectives described above, there are two quotient structures: The actual, $\dfrac{\mathbf{2}^{\omega}}{\mathrm{M}}$, and the potential, $\dfrac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}} \cong \dfrac{\mathbf{2}^{\omega}}{\mathbf{Fin}}$.

8.6.1 The actual quotient structure

There are $\aleph_0$ cosets in $\dfrac{\mathbf{2}^{\omega}}{\mathrm{M}}$, each of which corresponds to a prime, principal ideal in the derived set. These principal elements are the co-atoms of the algebra, with labels,

$\omega-\{1\}, \omega-\{2\}, \omega-\{3\}, \ldots, \mu$. Each corresponds to an open set of the unit interval from which one point has been removed. For example, $\omega-\{1\}$ could represent something like this: -



$$\omega-\left\{\tfrac{1}{2}\right\} \longleftrightarrow \left[0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right]$$

The cosets of $\dfrac{\mathbf{2}^{\omega}}{\mathrm{M}}$ can be written: $\mathrm{M}+(\omega-\{1\}), \mathrm{M}+(\omega-\{2\}), \mathrm{M}+(\omega-\{3\}), \ldots$; $\mathrm{M}=\mathbf{P}(\mu)$ is itself the zero of this quotient algebra, and $\mathrm{M}+\mu=\mathrm{M}$.

The collection of co-atoms comprises a Boolean space, and is specifically the Stone space of the Boolean algebra that is the dual to the Derived set. It is the dual algebra generated by taking the co-atoms as generators.[22]

8.6.2 The potential quotient structure

It is a standard result that $\dfrac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}} \cong \dfrac{\mathbf{2}^{\mathbb{N}_\lambda}}{\mathbf{Fin}}$ is non-atomic (see below); we may also show that it has $\mathbf{2}^{\omega}$ cosets. The cosets may be written, $\mathbf{Fin}+x$, where x is an infinite subset of the actually infinite skeleton. We write this, $x \subset \omega$. The cosets of $\dfrac{\mathbf{2}^{\mathbb{N}_\lambda}}{\mathbf{Fin}}$ are prime ideals arising from subtracting real numbers from the unit interval. They may be represented as $\mathbb{I}-\{\xi\}$ where $\mathbb{I}$ is the unit interval and ξ is a real number, where there are $2^{\aleph_0}$ reals. So there are $2^{\aleph_0}$ prime ideals, and, dually, $2^{\aleph_0}$ ultrafilters, each corresponding to a real number. The dual algebra of $\dfrac{\mathbf{2}^{\mathbb{N}_\lambda}}{\mathbf{Fin}}$ is the quotient algebra of all ultrafilters in $\mathbf{2}^{\mathbb{N}_\lambda}$ factored through by **Cofin.** $\dfrac{\mathbf{2}^{\omega}}{\mathbf{Cofin}}$ is the actually infinite collection of all real-number generators. The atom $\{\lambda\}$ belongs to this collection. I shall call the collection of all such ultrafilters the **boundary** of the Cantor set. The boundary comprises all ultrafilters with representation, $\text{filter}(\xi)$, where $\{\xi\}\subset 2^{\mathbb{N}_\lambda}$, but $\{\xi\}\notin 2^{\mathbb{N}_\lambda}$ together with all prime ideas, with representation $\mathbf{P}(\omega-\{\xi\})\subset 2^{\mathbb{N}_\lambda}$. The collection of ultrafilters is also called the collection of **generic ultrafilters**.

[22] This conforms to the Stone Representation Theorem.

8.6.3 Result, the atomless quotient algebra

The quotient algebra $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}} \cong \frac{\mathbf{2}^{\omega}}{\mathbf{Fin}}$ is atomless.

Proof

Let $[x] \neq [0]$ denotes any equivalence class in $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$; that is $[x] \in \frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$. Then $x \in \mathbf{2}^{\omega}$ is an infinite set. Since x is an infinite set it contains a countable number of infinite subsets that can be arranged in a chain. That is, there is also an infinite set $y \subseteq x$ such that $x - y$ is infinite. Since y is infinite $y \neq 0_{\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{(\omega)}}}$. Since $y \subseteq x$ we have $[y] \leq [x]$. Since y is infinite $x + y \notin \mathbf{2}^{<\omega}$; hence $x \not\equiv_{\mathbf{2}^{(\omega)}} y$ and $[x] \neq [y]$. So we have $0_{\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{(\omega)}}} < [y] < [x]$. That is, $[x]$ is not an atom. By generalisation, no equivalence class $[x] \neq 0 \in \frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ can be an atom. In $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ the equivalence classes are formed by taking any infinite set x and adding to that set any finite set (that is, any set $r \in \mathbf{2}^{(\omega)}$; that is, $[x] = x + \mathbf{2}^{(\omega)}$.

Remark (+)

If we replace $\frac{\mathbf{2}^{\omega}}{\mathbf{Fin}}$ in this argument by the atomic $\frac{\mathbf{2}^{\omega}}{\mathrm{M}}$ then M is a maximal element in a chain and so stops the crucial line the argument.

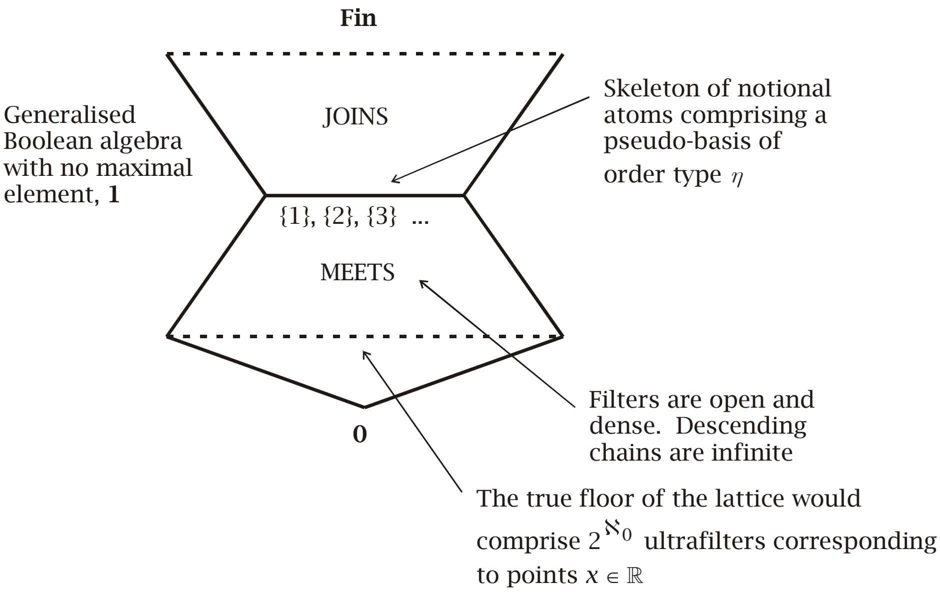
8.7 Complimentarity and collapsing cardinals

The paradoxical situation complimentarity requires further clarification. We have two structures:

Fin The set of all finite subsets of ω. Isomorphic to $\mathbf{2}^{<\omega}$ with cardinality $\aleph_0$.

$\mathbf{2}^{\mathbb{N}_\lambda}$ The Derived set. Isomorphic to the Cantor set, $\mathbf{2}^{\omega}$ and with cardinality $2^{\aleph_0}$.

The Derived set is the completion of **Fin** but <u>from within **Fin**</u> this is <u>not known</u>. **Fin** perceives its notional atoms as comprising a pseudo-basis for the continuum, so that there are joins above this basis and meets below it. So **Fin** extends in both directions to up-sets (ideals) and down-sets (filters). It perceives the filters as dense and open, so that within the filters there are infinitely descending chains. It perceives the pseudo-basis as a set of order-type η, which is the order-type of the rationals.



If the psuedo-basis could be completed, then it would have an actually infinite collection of atoms, but the "point at infinity" here representing the neighbourhood of 1 in the unit interval, $\{\lambda\}$, would still not be included in this collection, as it is not homogeneous with it. So from within **Fin** the continuum has a basis formed by adding one atom to a complete basis of ω atoms, and there is no bijection within **Fin** between this fictional basis of ω atoms for **Fin** and the complete skeleton for the unit interval. Hence, **Fin** perceives the skeleton of the continuum has having ω_1 atoms corresponding to its 2^{ω} ultrafilters; it sees the atoms as points and so accepts the Continuum Hypothesis, $2^{\omega} = \omega_1$. It perceives the ultrafilters as real numbers, that is, points of the arithmetical continuum.

The Axiom of Completeness allows us to complete the skeleton of the continuum to provide a basis of order-type $\omega + 1$ atoms. This basis for the Derived set is now orthogonal, meaning there are no meets of atoms. Since there are $\omega + 1$ atoms in the basis, and 2^{ω} points of the continuum, the Derived set perceives the basis as a set of sets of real numbers[23], and not as real numbers themselves. So the atoms are populated with real numbers rather than equated with them.

8.8 What is a real number?

It is accepted in first-order set theory that a real number corresponds to an infinite subset of ω. There are 2^{ω} such subsets.

[23] As sets of sets are added, this confirms that the Axiom of Completeness is second-order. It shows that, from this perspective, we cannot create a categorical structure for the continuum without second-order properties. Having a countable dense subset is interpreted as a second-order property, that is affirming the existence of sets of sets.

Let $x = \{x_1, x_2, x_3, ...\}$ be one such subset. However, consider carefully what this means. A set is a disjunctive list of its members corresponding to a join of singleton sets: -

$$x = \{x_1, x_2, x_3, ...\} = \{x_1\} \cup \{x_2\} \cup \{x_3\} \cup ... \qquad \leftrightarrow \qquad \{x_1\} \vee \{x_2\} \vee \{x_3\} \vee ...$$

When we consider **Fin** we identify a dense subset of the continuum and label its notional atoms as $\{1\},\{2\},\{3\},...$ this corresponds to some linear ordering of the dense subset, though it does not have a least element and is therefore not a well-ordering. Thus it is only a notional floor of the lattice and a pseudo-basis for the topology on the continuum. This means that intersections of these sets exist as do their correspondent lattice meets: -

$$\{1\} \cap \{2\} \cap \{3\} \cap ... \leftrightarrow \{1\} \wedge \{2\} \wedge \{3\} \wedge ...$$

Thus, in **Fin** there is a collection of 2^{ω} ultrafilters corresponding to infinite meets in the lattice. It is these that we identify with the real numbers. To summarise: -

1. In the Derived set we identify real numbers with infinite subsets of ω corresponding to actually infinite joins in the lattice.
2. In **Fin** we identify the real numbers with ultrafilters corresponding to potentially infinite meets of its notional atoms.

Strictly speaking this could be characterised as really confusing, because in one context we talk of joins and the other we talk of meets. We have to be clear as to what we mean: -

1. In the Derived set real numbers are equated with the principal elements of ideals corresponding to infinite joins of atoms. Real numbers in the Derived set are disjunctive sets. But not all joins correspond to prime ideals. A prime ideal is characterised by a principle element of the form $\omega - \{n\}$ where $n \in \mathbb{N}$, together with μ. There $\aleph_0$ prime ideals. There are 2^{ω} ideals in the Derived set corresponding to real numbers
2. In **Fin** real numbers are equated with ultrafilters, which are non-principal, but correspond to infinite meets of notional atoms. Real numbers generated by **Fin** are conjunctive sets. There are 2^{ω} such ultrafilters. Every ultrafilter corresponds to a real number generator.

8.9 The boundary

Another difference between **Fin** and the Derived set concerns the differing ways in which these two structures interpret the **boundary** between **Fin** and **Cofin** of the Derived set. From within **Fin** there is no filter of cofinite subsets; **Fin** has no knowledge of **Cofin**. **Fin** perceives that there is a boundary that lies wholly outside **Fin** to which it imagines that its ultrafilters are forever striving to reach. It interprets the boundary as populated with atoms corresponding to these ultrafilters, and concludes that if the ultrafilters could be completed there would be 2^{ω} of them, each defining a point of the boundary.

From the perspective of the Derived set, the boundary between **Fin** and **Cofin** lies wholly within the Derived set. The Derived set has a complete skeleton of $\aleph_0$ atoms, and $\aleph_0$ co-atoms. By recursive, potentially infinite, processes of addition of finite elements to finite sets we can never reach an infinite set, and by subtraction of finite elements from infinite sets we can never descend to a finite set; hence, the Derived set perceives that the boundary between **Fin** and **Cofin** is a **gap**. The gap can never be reached from within **Fin** by any potentially infinite process of joins (unions) of atoms, creating subspaces of the continuum, or by meets (intersections) of co-atoms. Neither the joins of atoms nor the meets of co-atoms can define points; they define only subspaces. Hence, while there are 2^{ω} points on the boundary these all belong to a gap that can never be reached from within **Fin** or **Cofin**. By taking joins and meets of actually infinite ω atoms we reach the boundary as a part of the Derived set.

8.10 Canonical extension of Fin

8.10.1 Lemma, canonical extension

If *B* is a canonical extension of a Boolean algebra *A*, then the distinct atoms in *B* are precisely the infima of the distinct ultrafilters in *A*. (For the proof see Givant and Halmos [2009], Lemma 1, Chapter 23, p.195.)[24]

Remark

There are also existence and uniqueness theorems for the canonical extension. The canonical extension is atomic, complete and compact.

Fin is a generalised Boolean algebra rather than a Boolean algebra, but, subject to the Prime Ideal theorem it may be extended to a maximal ideal, M, which is a Boolean algebra. The lemma tells us the atoms of this canonical extension are the infima of the ultrafilters. Since there are $\mathfrak{c} = 2^{\aleph_0}$ ultrafilters in **Fin** and $\omega = \aleph_0$ ultrafilters in M, the canonical extensions of these two structures are not identical.

1. Canonical extension of **Fin**

Let $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\aleph_\lambda}}$ represent the algebra with the skeleton comprising of all points $\xi \in \mathbb{R} \cap [0,1]$. The ultrafilters of the potentially infinite lattice, $\mathbf{2}^{<\omega}$, correspond to the real numbers $\xi \in \mathbb{R} \cap [0,1]$. The atoms of $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\aleph_\lambda}}$ are the real numbers of the

[24] I assume here, without proof, that this applies also to a generalised Boolean algebra, which lacks a topmost element, **1**.

continuum corresponding to the infima of the ultrafilters in **Fin**. $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\mathbb{N}_\lambda}}$ is the canonical extension of **Fin**. We will call $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\mathbb{N}_\lambda}}$ the **Canonical extension**.

2. The canonical extension of M.

This is the derived set $\mathbf{2}^{\mathbb{N}_\lambda}$; the ultrafilters of M are the atoms of $\mathbf{2}^{\mathbb{N}_\lambda}$.

The Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, is embedded in the **Canonical extension,** $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\mathbb{N}_\lambda}}$ of **Fin**. $\{\lambda\}$ is an atom of both the Derived set and the Canonical extension ; and μ is a co-atom of both; otherwise, the atoms of $\mathbf{2}^{\mathbb{N}_\lambda}$ are represented by the skeleton $\mathbb{N}_\lambda = \mu \cup \{\lambda\}$, and the atoms of $\mathbf{2}^{\mathbb{I}} = \mathbf{2}^{2^{\mathbb{N}_\lambda}}$ by $\left\{\{\xi\} : \xi \in \mathbb{R} \text{ iff } \operatorname{filter}(\xi) \text{ is an ultrafilter in } \mathbf{Fin}\right\}$. We may show that $\{\lambda\}$ is an ultrafilter of **Fin** because $\{\lambda\}$ is the meet of all notional atoms in **Fin**; when the set of notional atoms in **Fin** is completed, we obtain the actually infinite $\{\lambda\}$. This also confirms that $\{\lambda\} = \{\{\mathbb{N}\}\}$.

It is the Axiom of Completeness that asserts the existence of a least upper bound to every Dedekind cut, or equivalently, in the Nested Interval Theorem, of a real number that is the content of every member of an actually infinite sequence of nested intervals. Denoting such a sequence by $\operatorname{filter}(\xi)$, we see that we can place every ultrafilter of **Fin** in one-one correspondence with a point of the arithmetical continuum, $\xi \in \mathbb{R}$.

8.11 "Identity up to isomorphism" and the prolixy of set theory

It is an error to assume that the Cantor set has an unambigous structure, as the prolixy of solutions to the continuum question demonstrates. If it is possible to have any solution consistent with ZFC to the equation, $\mathbf{2}^{\omega} = \omega_\kappa$, then it follows that the structure of $\mathbf{2}^{\omega}$ is under-determined.

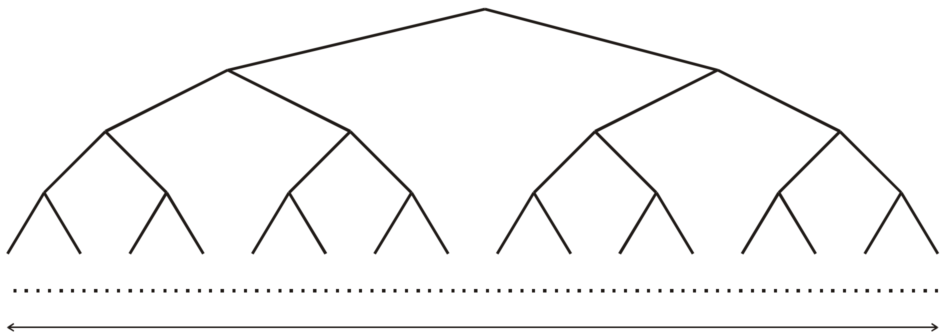
It is not difficult to see why this must be the case. The axioms of ZFC are insufficient to determine the answer to the problem of the continuum. The power set axiom generates <u>ambiguously</u> in one-step the totality of all functions, $\omega \to \mathbf{2}$. The other axioms, particularly the replacement axiom, provide tools for generating this totality from below, but there is an insufficient overlap between the two collections of tools, so the structure is under-determined.

Already, in the definition of the power set axiom on $\omega \to \mathbf{2}$, and in Cantor's justly famous proof that the cardinality of $\mathbf{P}(\omega) = 2^{\omega}$, we have the assumption that a <u>totality of all functions</u> $\omega \to \mathbf{2}$ exists. In other words, we assume the existence of an actual infinity. If the terms "all" or "totality" in $\omega \to \mathbf{2}$ refer only to a potential infinity, then the collection $\omega \to \mathbf{2}$ is fully determined from below by ordinal exponentiation. That is to say, if there is no actual totality of all functions, but the totality of all functions refers only to the possibility of generating one more function in a recursive sequence, then the answer to the cardinality question is just $2^{\omega} = \operatorname{card}(\mathbb{N}) = \aleph_0$. Granted only potential infinities, then the whole apparatus of transfinite ordinals: $\omega_0, \omega_1, \ldots$ becomes superfluous, for then there is only one infinity – the potential infinite – and only one infinite cardinality – the cardinality of the potential infinite, and the question of the continuum is unambiguously solved.

We have already argued that the concept of the potential infinite is inadequate to provide a suitable model of the continuum for science, which does require the actual infinite and the preconceived totalities of all real numbers and all functions: $\omega \to \mathbf{2}$ and $\omega \to \omega$ giving rise to Cantor space, 2^ω and Baire space ω^ω, which may both be taken as representing sets of the collection of all real numbers. It was the struggle of C19th analysis to demonstrate the need for the actual infinite.

However, the question of ambiguity immediately surfaces, for then it is not possible to say what a **topologically invariant** structure to the Cantor set, 2^ω, would look like until further axioms are provided to define that structure. Until that is done $\mathbf{2}^\omega$ is a mere symbol representing the image of the power set operation on the totality of all mappings: $\omega \to \mathbf{2}$ and nothing more.

The ambiguity of $\mathbf{2}^\omega$ can also be seen from consideration of its tree structure.



$\mathbf{P}\,(\omega) = \mathbf{2}^\omega$

Here the dots represent the fact that the branches of the tree are infinite in length. The power set axiom determines that the width of the tree at the base, after the branches become actually infinite, is the ordinal $\mathbf{2}^\omega = \mathbf{P}(\omega)$, whose cardinal size $2^{\aleph_0}$ is defined to be the size of the continuum: $\mathfrak{c} = 2^{\aleph_0}$. ZFC also determines, through Hartog's theorem, that there is a sequence of increasing ordinals: $\omega_0, \omega_1, \ldots, \omega_\kappa, \ldots$ though the upper end of this scale is also ambiguous. ZFC does not determine how to match the ordinal $2^\omega = \mathbf{P}(\omega)$ with this scale. It is determined that this tree has a potentially infinite sub-tree; the structure denoted $2^{<\omega}$ which is isomorphic to **Fin**, the collection of all finite subsets of ω.

Looking at the tree one might be tempted to say that the length of the branches in $2^\omega = \mathbf{P}(\omega)$ is also determined, for after all they are infinite. But it is consistent with ZFC that $2^\omega = \omega_2$, meaning that before we reach the terminal stage of this tree, we have generated a distinct set of ordinal length ω_1, either as a branch or as a level. The dots in the above diagram represent a gap between the finite part of the tree and the actually infinite part. Therefore, above every terminal point in the tree, where the limit has been attained, there is an actually infinite tree; furthermore, this actually infinite tree is a replica of the tree as a whole. Therefore, it is not inconceivable, that the branches of the tree must actually take length ω_1 in order for the totality of all branches, $2^{\omega} = \mathbf{P}(\omega)$, to be generated in the limit. Similarly, if $2^{\omega} = \omega_2$ then at some stage of the generation of the tree its width must attain ordinal ω_1, where $\omega_0 < \omega_1 < \omega_2$. Thus, in conclusion, the axioms of ZFC are consistent with many interpretations of the structure of the Cantor set.

To further clarify this point, in set theory the Cantor set $2^{\omega} = \mathbf{P}(\omega)$ is primarily defined as the totality of all functions: $\omega \to 2$. This appears to unambiguously structure the Cantor set as having only branches at the limit of maximal length ω and only at the limit a width of size $\omega_1 = 2^{\omega}$ embedded within which is an antichain of length ω, which corresponds to the skeleton of the dense partition of the continuum. All of this would imply the Continuum hypothesis. However, in set theory the Cantor set is a structure embedded within the proper class of all sets whatsoever, so the possibility exists that in the first instance there is some partial order lying <u>outside</u> the Cantor set that serves to add additional structure to the Cantor set and thence to the continuum. For example, a partial order corresponding to a forcing: $\mathbf{C} : \omega \times \kappa \to 2$ would achieve this. Assuming that this forcing also satisfies a condition known as the countable chain condition, this then implies $2^{\omega} = \omega_{\kappa}$. This in turn would demonstrate that $2^{\kappa} = 2^{\omega}, \kappa > \omega$, contradicting the <u>weak continuum hypothesis</u>, $2^{\kappa} > 2^{\omega}, \kappa > \omega$, whereupon the Cantor set, $\omega \to 2, 2^{\omega}$ becomes isomorphic to the structure, $\omega \times \kappa \to 2, 2^{\kappa}$, so we see that the structure of the Cantor set in ZFC is highly ambiguous.

Thus it has been concluded in the literature that the question depends on the adoption of further axioms; it is implicit in this discussion that these axioms will supply the missing structure to the Cantor set. But the discussion is constrained firstly by the assumption that only first-order axioms shall be considered, and secondly by the manifest problem that no one axiom can really be said to be a better candidate than another. If the question could become purely empirical then some progress by that approach might be possible; however, the question of how to experimentally test a model of the arithmetical continuum has not yet been considered.

Nonetheless, it emerges that there is a pre-existing structure for the Cantor set provided by the second-order Axiom of Completeness, which has some historical claim to importance, and furthermore, given this Axiom, we have already gone a long way in this paper to demonstrating an unambiguous structure to the Cantor set. Understanding that the arithmetical continuum is generated in the following three stages:

1. A potentially infinite dense subset provides a basis for recursive sequences that generate real numbers; these sequences are functions from $\omega \to \mathbf{2}$, or equivalently,

$\omega \to \omega$. This dense subset is usually designated $\mathbb{Q}$, but as this structure is equinumerous to $\mathbb{N}$, it may also be taken to be $\mathbb{N}$. This is the separation property: every set of cardinality continuum has a dense subset.

2. Subject to the Heine-Borel theorem, this potentially infinite dense basis is extended to an actually infinite partition of the continuum, which becomes the skeleton of a complete Boolean algebra. Real number generators are equated with the collection of all ultrafilters in the set **Fin**.
3. The arithmetical continuum is then generated as the derived lattice, Boolean algebra, of the actually infinite skeleton of the continuum. This generates the continuum as a collection of prime ideals which correspond to functions $\omega \to \mathbf{2}$, that is to sequences enumerating the binary expansion of a real number. The conception is further completed by the assumption that each binary sequence converges on a unique real number, which is a point of the arithmetical continuum.

Subject, then, to the second-order Axiom of Completeness, the Cantor set has an unambiguous structure, and the question of the size of the continuum becomes: what is the cardinality of the structure $2^{\mathbb{N}_\lambda}$?

9. The algebraic numbers

9.1 Polish spaces

This is the family of separable, complete metric spaces and includes the Euclidean spaces, Cantor space and Baire space. Those that we are particularly concerned with are Cantor space, denoted 2^ω, Baire space denoted ω^ω, the real line (arithmetical continuum), denoted $\mathbb{R}$, and the unit interval, denoted $\mathbb{I} = [0,1]$.

It is possible to demonstrate: -

1. A standard injection of Baire speace into Cantor space
2. A standard binary expansion of real numbers
3. A standard surjection of the Cantor space into the unit interval.[25]

Together these ensure that that the Cantor space and Baire space may be taken as representatives of the real numbers in the unit interval, in that (a) they have the same cardinality, which is continuum, and (b) they possess the fundamental property of being linear orders that are complete and separable. This enables us to switch between all three representations, and to transfer results that are more naturally demonstrated in one context

[25] This is derived primarily from the treatment in Levy, Azriel [2002].

to another. One particularly important example of this process concerns the algebraic numbers, where the result that the set of all algebraic numbers is countably infinite is proven in the context of Baire space, but we wish subsequently to transfer it to Cantor space, which is the Boolean model of (some finite part) set theory that we are working with. This is achieved automatically by the existence of the bijection that maps the real numbers to their standard binary expansion.

However, the differences between the three representations are also significant and are frequently ignored. In particular, both Cantor space and Baire space are discrete and totally disconnected spaces. Since $\mathbb{R}$ is connected it is not homeomorphic to either Cantor or Baire space. This difference is crucial. It means that Cantor space is not homeomorphic to the unit interval, and the two sets are not identical. Hence, to become a true representative of the unit interval we must add to the notion of the Cantor set some additional properties. This has been the whole theme of this monograph. I will further show below [Section 10] that in order to be a model of the unit interval the skeleton of the Cantor set in the form of the Derived set must be treated as a collection of atoms and co-atoms, where the atoms are boundary points, and the co-atoms converge in the limit on extension points. None of this is conveyed by the mere image of the Cantor set, $2^{\omega} = \{0,1\}^{\omega}$, which is an ambiguous structure when it comes to treating it as a model of the continuum. Levy [2002] also draws our attention to the fact that "Cantor and Baire spaces are "dimensionless", unlike the real line $\mathbb{R}$." It is this that points to the absence of the primitive notion of extension, which shall be explored below [Section 10.4]. Cantor and Baire space are mere collections of boundary points with null measure.

9.2 Algebraic numbers

Let $\mathbb{Z}[t]$ be the ring of polynomials in one indeterminate *t*, with integer coefficients $a_k \in \mathbb{Z}$; these may be positive or negative integers. The polynomial has representation: -

$f = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$ where $a_n \neq 0$.

If $a_n \neq 1$ then the polynomial is said to be monic. Let α be a root of *f*. Suppose that for all $k < n$ there is no other polynomial for which α is a root, then *f* is said to be a minimum polynomial for α. It is customary to indicate a minimum polynomial by denoting it $m = m(t)$.

Then an algebraic number is a root of a non-zero minimum polynomial, $m = m(t)$, of one variable with integer coefficients.

We wish to relate the algebriac numbers to the ideal of all finite subsets of the Cantor set, which is denoted $\mathbf{Fin} = \mathrm{F}(\omega) \cong 2^{<\omega}$. We will show that every algebraic number corresponds to some lattice point in $\mathbf{Fin} = \mathrm{F}(\omega) \cong 2^{<\omega}$. Firstly let us prove that the set of all algebraic numbers is countably infinite.

9.2.1 Cantor's theorem

The set of real algebraic numbers is denumerable

Proof

The **weight of a polynomial,** $f(x)=\sum_{i=0}^{n}a_i x^i$, is defined to be the number $n+\sum_{i=0}^{n}|a_i|$.

For any given weight there are only a finite number of polynomials having that weight. These may be arranged lexicographically, for example, by listing them by order of n first, and then in order of a_0. Every non-constant polynomial has a weight of at least 2. List all polynomials in order of weight followed by the lexicographic order. Every polynomial of degree 1 or more appears in the list just once. Each polynomial has a finite number of real zeros, so a denumerable list of the zeros may be derived from the list of the polynomials. The list is infinite because $\mathbb{Q}$ is a subset of it.

This result already settles this issue. However, to be more explicit we proceed as follows. So far we have considered the atoms of the skeleton of the real line to be represented by a sequence: $\mathbb{N}_\lambda=\{\{1\},\{2\},\{3\},\ldots,\{\lambda\}\}$. But the members of this set are only labels, and the potentially infinite part of this, $\mathbb{N}\cong\{\{1\},\{2\},\{3\},\ldots\}$, being non-atomic, is a collection of notional atoms. To display the connection with the algebraic numbers, we choose a representation in which the notional atoms are ordered pairs of numbers, $\langle j,n\rangle$ where $j\in\{0,1\}$ and $n\in\mathbb{N}$. We think of this as an encoding of an integer coefficient of $\mathbb{Z}$. Thus the coefficients $3\in\mathbb{Z}$ and $-4\in\mathbb{Z}$ correspond to ordered pairs $\langle 0,3\rangle$ and $\langle 1,4\rangle$ respectively. We regard the atoms as externally ordered in this model, so we require the Axiom of Choice to bring this about. In this way we may map a set of atoms unambiguously to a polynomial in $\mathbb{Z}[t]$. For example, the sequence of atoms, $\langle\langle 0,3\rangle,\langle 1,4\rangle,\langle 0,0\rangle,\langle 1,2\rangle\rangle$ maps to the polynomial $3-4t-2t^2$. Looking at $\langle\langle 0,3\rangle,\langle 1,4\rangle,\langle 0,0\rangle,\langle 1,2\rangle\rangle$ as the ordered meet of atoms, $\langle\langle 0,3\rangle,\langle 1,4\rangle,\langle 0,0\rangle,\langle 1,2\rangle\rangle=\langle\ \langle 0,3\rangle\wedge\langle 1,4\rangle\wedge\langle 0,0\rangle\wedge\langle 1,2\rangle\ \rangle$, we have an embedding of minimum polynomials of the ring $\mathbb{Z}[t]$ within the lattice of finite subsets of Baire space. That the meet exists follows from the fact that the lattice is non-atomic. To each of these mimimum polynomials corresponds a set of algebraic numbers; let ∂m denote the degree of the minimum polynomial $m(t)$ then there are $<\partial m$ algebraic roots at the correspondent lattice point. This is an surjection of the polynomial ring $\mathbb{Z}[t]$ into the set of finite subsets of Baire space.[26]

[26] The relationship between minimum polynomials and their distinct roots is clarified in Galois theory. However, in this context we only require that all roots correspond to some minimum polynomial or other. Therefore, the complexities of the Galois theory and the need to define the concepts of normal and separable field extensions need not be of concern here.

The encoding on which this is based, $\langle j,n\rangle$ where $j \in \{0,1\}$ and $n \in \mathbb{N}$, represents each polynomial $\mathbb{Z}[t]$ as corresponding to a member of Barie space, ω^ω. Employing the principle that Baire space has as many lattice points as Cantor space, and using the standard binomial representation of any number $n \in \omega$, we see that to every algebraic number there corresponds to a meet in the potentially infinite part of the Cantor set: $\text{Fin} = F(\omega) \cong 2^{<\omega}$.

10. Decomposition of the skeleton

10.1 Measure and Category on the arithmetical continuum

We require some background theory concerning perfect, meagre and null sets. All of these results follow from the Axiom of Completeness. Given a set *S*, a point *x* is an **accumulation point** (also known as **limit point**) of *S* if every open interval that contains *x* also contains infinitely many points of *S*. The set of accumulation points of *S* is called its **derived set**. A **perfect set** is a set that is its own derived set. So a perfect set is equal to its own closure. There is a construction of the Cantor set, $\mathbf{2}^\omega = \{0,1\}^\omega$, that involves the systematic removal of segments of the closed unit interval. This displays it as the **Smith-Volterra-Cantor set** $SVC(3)$. The length of any interval *I* is denoted $|I|$. The **measure** of a set, *S*, denoted $\mu^*(S)$, is defined in such a way that the measure of an interval is equal to its length: $\mu^*(I) = |I|$; that is, if $I = [a,b]$ or $I = (a,b)$ then $\mu^*(I) = b - a$.[27] We do not here go into the details of the definition of measure; it is sufficient to know that the measure of an interval is equal to its length, which is also called its **outer content**. It is a **result** that the Cantor set is perfect, nowhere dense and has outer content equal to zero. A set $A \supset \mathbb{R}$ is called a **nullset** (or a **measure zero set**) if for each $\varepsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \varepsilon$. Singletons are nullsets and any subset of a nullset is a nullset. Any countable union of nullsets is a nullset. A set is said to be **first category** or **meagre** if it can be represented as a countable union of nowhere dense sets. A subset of *R* that cannot be so represented is said to be of **second category**. **Baire's theorem** states that the complement of any set of first category on the line is dense. No interval in $\mathbb{R}$ is of first category. The intersection of any sequence of dense open sets is dense. Measure zero sets and sets of first category are "small", but in different senses. Oxtoby [1980] remarks, "A nowhere dense set is small in the intuitive geometric sense of being perforated with holes, and a set of first category can be "approximated" by such a set. A set of first category may or may not have any holes, but it always has a dense set of gaps. No interval can be represented as the union

[27] I am using μ to denote the principal element of the prime ideal of all finite sets, $\mathrm{M} = \mathbf{P}(\mu)$, $\mu = \{\lambda\}'$ and μ^* to denote the measure of a set.

of a sequence of such sets." Neither class (measure zero, first category) contains the other; "null and meagre describe "smallness" in two different ways." (Bartoszynski and Judah [1995]) The symbols $\mathcal{N}$ to denote the **ideal of all null sets** in $\mathbb{R}$. and $\mathcal{M}$ to denote the **ideal of all meagre sets** in $\mathbb{R}$ are standard.

10.2 The meagre-null decomposition of the real line

10.2.1 Theorem, meagre-null decomposition

The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.

Corollary

Every subset of the line can be represented as the union of a nullset and a set of first category.

Oxtoby [1980] remarks that "A similar construction can be done in 2^ω." This is correct, and we shall presume that any construction performed in ω^ω is equally valid for 2^ω. The commentary below will clarify this.

Proof of the theorem

There exist sets $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$

Let $\langle q_n : n \in \omega \rangle$ denote an enumeration of rationals according to some rule. Let $I_{i,j} = \left(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n} \right)$.

Let $G_j = \bigcup_{i=1}^{\infty} I_{i,j}$ $(j = 1,2, \ldots)$ and $B = \bigcap_{j=1}^{\infty} G_j$.

For any $\varepsilon > 0$ we can choose j so that $\frac{1}{2^j} < \varepsilon$.

Then $B \subset \bigcup_i I_{i,j}$ and $\sum_i |I_{i,j}| = \sum_i \frac{1}{2^{i+j}} = \frac{1}{2^j} < \varepsilon$. Hence B is a nullset.

However, G_j is a dense open subset of $\mathbb{R}$ being the union of a sequence of open intervals; it also includes all rational points. Hence its complement $\overline{G_j}$ is nowhere dense. Hence, $A = \overline{B} = \bigcup_j \overline{G_j}$ is meagre.

That is, $A \in \mathcal{N}$ and $B = \mathbb{R} - A \in \mathcal{M}$.

The essential idea of the proof is to trap every rational number inside an individual open interval which can be shrunk to a size as small as one pleases. These open sets comprise intervals that at any finite stage of the process overlap and cover the unit interval. In the limit they shrink to null sets and uncover a meagre, boundary set that has measure 1.

In the proof every rational is enclosed in an open set of size $\frac{1}{2^j}$ where $j \to \infty$. What we are doing is enclosing every rational number within an open interval G_j of a size that can be made arbitrarily small, $\mu^*(G_j) < \varepsilon$. This then gives us a family of a countably infinite dense open subsets of $\mathbb{R}$ and A is the union of these such that $\mu^*(A \cap [0,1]) = 0$. Then its complement $B = [0,1] - A$ is a countably infinite set of (closed) nowhere dense subsets of $[0,1]$, and so meagre; it measure is $\mu^*(B \cap [0,1]) = 1$.

The family of open intervals, G_j, has cardinality $\aleph_0$, but both A and B are unions of members of these families; the cardinality of each G_j is continuum, so the cardinality of both the A and B sets is continuum. This decomposition demonstrates that a meagre set may have positive measure. Thus we have: -

A	A null set	$\mu^*(A) = 0$
B	A meagre set	$\mu^*(B \cap [0,1]) = 1$

The decomposition, $\mathbb{R} = A \cup B$, where $B = \mathbb{R} - A$, is a decomposition of the skeleton of the one-point compactification of the real line. Sets A and B are themselves collections of $\mathfrak{c} = 2^{\aleph_0}$, continuum many points. Set A may be thought of as a collection of subsets, each of which is "centred" one some unique rational number; we shall call these **clusters**. This is what arises from the decomposition. However, the members of these clusters are real (specifically, irrational) numbers that do not belong to the meagre set *B*. The members of *B* are real numbers not belonging to the null set *A*. The irrational numbers may be sub-divided into (a) algebraic irrationals, and (b) transcendental reals. Since algebraic numbers correspond to lattice points of **Fin**, this entails that the transcendental numbers may be classified as: -

(a) Transcendental reals not belonging to the meagre set *A* in the above decomposition.
(b) Transcendental reals not belong to the null *B* set in the above decomposition.

10.2.2 Definition, Cohen and amoeba reals

A **Cohen real** is a transcendental number not belonging to the meagre set *A*.
An **amoeba real** is a transcendental number not belonging to the null set *B*. Let $\mathbf{A}_\varepsilon = \{U : U \subseteq \mathbf{2}^\omega, U \text{ is open and } \mu^*(U) < \varepsilon\}$ (See Bartoszynski and Judah [1995].)

Then *U* is an amoeba real.[28]

[28] Concerning the difference between an amoeba and a random real, which are also generally characterised as real numbers not belonging to any null set. (Please refer also to section 14 on Forcing and generic sets.) Random forcing within the context of first order set theory (ZFC) is defined relative to an encoding of sequences that converge on random reals. These sequences are called Borel codes.

Cohen reals comprise clusters of reals each associated with a unique rational number, $q \in \mathbb{Q}$.

10.3 The decomposition of the one-point compactification

We continue to examine the properties of the actually infinite skeleton of the continuum: $\mathbb{N}_\lambda$. By an abuse of notation, through out this subsection we will allow $\mathbb{N}$ to denote the potentially infinite part of the skeleton. That is, here, $\mathbb{N}$ represents $\{\{1\},\{2\},\{3\},\ldots\}$. However, we shall also use $\mu = \{\{1\},\{2\},\{3\},\ldots\}$ where we wish to emphasise the relationship between this set and the prime ideal M which is the supremum of all families of sets in **Fin**. (μ in this sense should not be confused with $\mu^*(X)$ to denote the measure of a set.) Since function of the skeleton is to provide a dense subset of the real line, then it follows that the skeleton may be equally denoted $\mathbb{Q}_\lambda$, and we have $\mathbb{Q}_\lambda \sim \mathbb{N}_\lambda$. There is a standard bijection between $\mathbb{N}$ and $\mathbb{Q}$, and so we label the basis elements of the skeleton by atoms, $\{1\},\{2\},\{3\},\ldots,\{\lambda\}$, bearing in mind that these may also be paired off with an enumeration of rational coefficients, for example, $\{1\} \leftrightarrow \frac{1}{2}, \{2\} \leftrightarrow \frac{1}{3}, \{3\} \leftrightarrow \frac{2}{3}, \ldots$. So let us also think of each label as denoting a rational number. Therefore, we may visualise each atom as introducing a boundary into the unit interval $\mathbb{I} = [0,1]$ corresponding to a rational number. This boundary is a set of zero dimension and zero measure.

Borel codes

Every Borel subset of the continuum can be constructed from basic sets in countably many steps. The information about this construction can be stored in one real number. BOREL denotes the set of Borel codes.

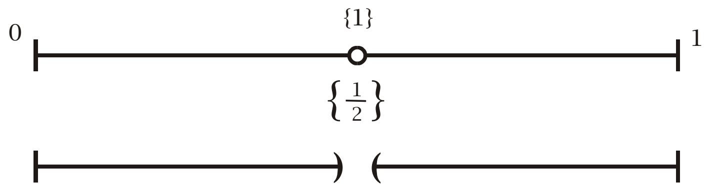
Random forcing is defined specifically as: -

Random forcing

$\mathbf{B}(\kappa) = \{[A]_{\mathcal{N}} : A \in \mathrm{BOREL}(2^\kappa)\}$ where $\mathcal{N}$ is the null ideal.

Amoeba forcing is not forcing in the sense in which random forcing with BOREL codes is, or Cohen forcing. Given Cohen forcing, $\mathbb{C} = \mathrm{Fn}(\omega,2)$, amoeba forcing is what is left over in the continuum once the Cohen reals and other boundary points are removed from it.

Random reals are a possible sub-category of smoeba reals. However, Cohen forcing does not specifically add them and it is the claim here that subject to the Axiom of Completeness the continuum has only Cohen forcing, denoted $\mathbb{C} = \mathrm{Fn}(\omega,2)$, that these add amoeba reals and no other reals. So in this model there are no random reals.



$$\omega-\left\{\tfrac{1}{2}\right\} \longleftrightarrow \left[0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right]$$

Suppose the first atom $\{1\}$ in the skeleton correspond to the boundary point $\dfrac{1}{2}$ in the unit interval $\mathbb{I}=[0,1]$. By removing this atom, we obtain the co-atom $\omega-\{1\}$; this corresponds to the open interval $\left[0,\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},1\right]$; it is open because the open interval $[0,1]$ is both closed and open in itself. We have for each atom: $\{n\}\in\mathbb{N}\leftrightarrow\{q\}\in\mathbb{Q}$; to each of these atoms there corresponds a cluster of Cohen reals in the meagre-null decomposition. The collection of all such reals is the A set of the above decomposition. Then we observe that the collection $\mathbb{N}=\{\{1\},\{2\},\{3\},\ldots\}\leftrightarrow\left\{\left\{\dfrac{1}{2}\right\},\left\{\dfrac{1}{3}\right\},\left\{\dfrac{2}{3}\right\},\ldots\right\}$ can no longer be taken as the complete skeleton of the real line; taken collectively it is both a meagre and null set corresponding to rational numbers that act as a basis for the A set, and provides no basis whatsoever for the B set. Consider the collection of coinfinite subsets of ω: $\omega-\{1\}$, $\omega-\{2\},\ldots,\ldots,\omega-\{\lambda\}$. These comprise the co-atoms of the Derived set. These act as a basis for the B set in the decomposition theorem. Since each $\{n\}\in\mathbb{N}=\{\{1\},\{2\},\{3\},\ldots\}$ correlates to a rational point, each $\omega-\{n\}\in\omega-\mathbb{N}$ is an <u>interval</u> in the unit interval $\mathbb{I}=[0,1]$. Furthermore, since each $\{n\}\in\mathbb{N}$ represents an isolated boundary point, then each $\omega-\{n\}$ represents an open interval in $\mathbb{I}=[0,1]$. So the skeleton of the arithmetical continuum <u>must</u> be decomposed into two collections: -

$\mathbb{N}_A$ The skeleton of atoms of the derived Cantor set of the continuum, which are boundary points corresponding to rational numbers, and collectively both a null and meagre set. This collection has measure $\mu(\mathbb{N}_A)=0$. It has canonical representation: -

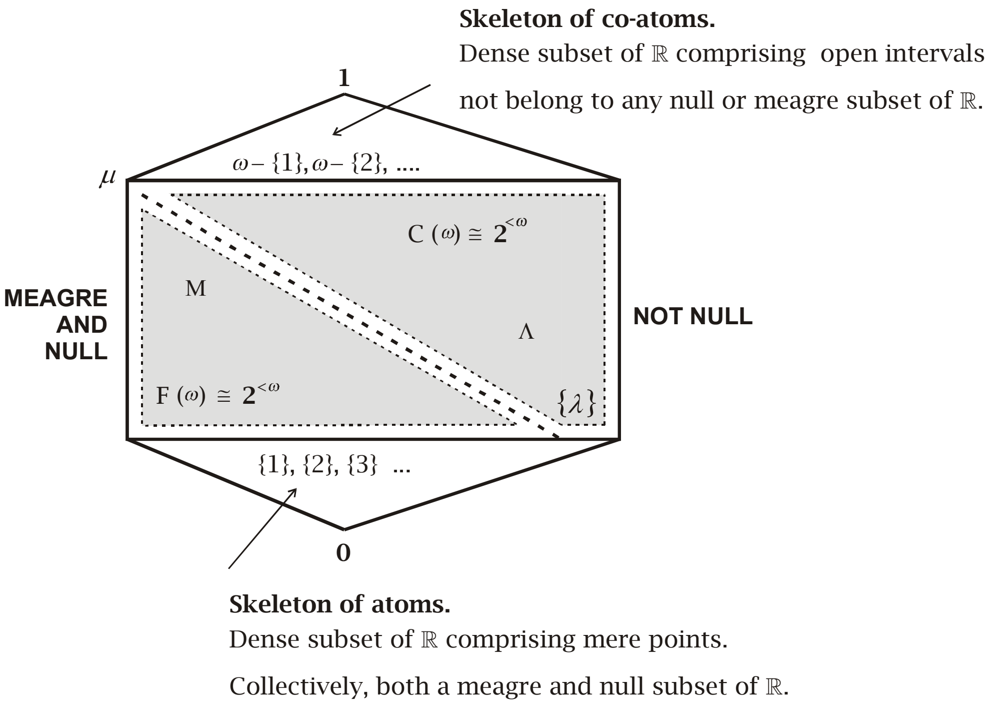
$\mathbb{N}_\lambda=\{\{1\},\{2\},\{3\},\ldots\}$. The atom $\{\lambda\}$, which represents the neighbourhood of 1, is not a member of this collection.

$\mathbb{N}_B$ The skeleton of co-atoms of the derived Cantor set of the continuum, which are open intervals and collectively neither null nor meagre sets. In the unit interval this skeleton has measure $\mu([0,1])=1$. It has canonical representation: -

$\omega-\mathbb{N}_\lambda=\{\omega-\{1\},\omega-\{2\},\omega-\{3\},\ldots\}$.

> The co-atom $\mu = \{\lambda\}' = \omega - \{\lambda\}$ is not a member of this collection and represents the neighbourhood of 0 in the interval $[0,1]$. It is this neighbourhood that has been perforated by a dense collection of points corresponding to the rational numbers in the skeleton $\mathbb{N}_A$.

By Baire's theorem, the complement of any set of first category on the line is dense. Hence the skeleton of co-atoms is dense in $\mathbb{R}$. So the Derived set of the one-point compactification of the real line has the following structure: -



We need to add some clarificatory remarks about the representation of a co-atom by, for instance, $\omega - \{1\}$. Taking $\omega = \{1,2,3,\ldots\}$, then $\omega - \{1\} = \{2,3,4,\ldots\}$ is <u>a mere collection of numbers</u>. There is nothing in $\omega - \{1\}$ to indicate that as a collection it represents an interval. It is the meagre-null decomposition of the line that <u>forces</u> us to treat $\omega - \{1\}$ in this way. If we do not regard $\omega - \{1\}$ as corresponding to an interval, then it is a meagre and null set, and there is no interval of positive measure in the entire structure.

10.4 Boundaries and extensions

The decomposition of the skeleton forces us to introduce a distinction between two kinds of points: boundaries and extensions.

10.4.1 Definition, point

A **point** is that part of the continuum that corresponds to a real number, $\xi \in \mathbb{R}$.

10.4.2 Definition, absolutely zero measure

I shall say that a point has **absolutely zero measure** if no collection of such points of whatever cardinality has positive measure.

10.4.3 Definition, boundary point or boundary

Boundaries are points of absolutely zero measure.

Boundaries correspond to what Aristotle says about points having no extension.

> Points are dimensionless. No continuum is composed of points. Points act as boundaries or limits only.[29] [Section 3.1]

Evidently, we cannot make a continuum out of boundary points. Note, that the Cantor set proves that such a set of cardinality $\mathfrak{c}$ exists, because the Cantor set is a set of $\mathfrak{c}$ points with zero measure.

10.4.4 Result (+)

Every collection of boundaries is meagre (first category in $\mathbb{R}$).

Proof

From Baire's theorem, no interval in $\mathbb{R}$ is first category. No collection of boundaries is an interval.

Since we cannot make a positive measure of such points, we must have another kind of point.

10.4.5 Extension points or extensions

Extensions are points of measure incommensurable with zero, but such that any collection of continuum many such points has finite measure.[30]

The exact measure is undetermined: there is continuum many points in any interval, so it is not possible to infer an exact finite measure from a collection of any cardinality. We cannot

---

[29] On the phenomenological continuum, boundaries are also fictional or idealised elements. However, on the arithmetical continuum we drop this property, because we have accepted into our science as coherent the notion of an actual infinity, and this entails that the continuum is a collection of points. So relative to the arithmetical continuum, the points are no longer fictional. The theory of the arithmetical continuum as a whole may be a fiction, so we transfer the property of being a fictional entity from the part (boundary point) to the whole (the arithmetical continuum).

[30] Extensions or extension points might have some claim to be called infinitesimals; however, they do not necessarily have the properties ascribed to them in the recent Robinson axiomatisation of infinitesimals and they obey the Axiom of Archimedes, so we shall not use that term.

compose an interval from any collection of boundaries; w infer the existence of extensions from the fundamental property of the Cantor set, that it is a totally disconnected, nowhere dense subset of the continuum which is first category in the continuum and of zero measure. Hence there are two collections: one of boundaries and the other of extensions. We may also work in the reverse direction by adopting the following principle or "axiom".

10.4.6 "Axiom of indestructibility of extensions"

> Let $\bigcap_{n\in\omega} I_n$ be an actually infinite sequence of nested intervals. If $\bigcap_{n\in\omega} I_n \neq \varnothing$, then $\bigcap_{n\in\omega} I_n$ is never a boundary; $\bigcap_{n\in\omega} I_n$ is always an extension.

Any interval on the continuum, open or closed, $[a,b],(a,b),\ldots$, self-evidently represents an extension in space. What the axiom of indestructibility of extensions says is that we can never transcend the class of extensions to obtain a point of no extension whatsoever. In this way we seek to embody the following aspect of the phenomenological continuum: -

> Space is composed only of space; subdivision of the continuum generates only another continuum. [Section 3.1]

From this statement we may say that a synonym of "extension" would be "space"; but whereas "extension" implies something linear, or of one-dimension, space implies more than one dimension. We also place the term "axiom" in this definition within scare quotes. This is because it is not really an axiom, but a theorem consequent on the second-order Axiom of Completeness. We have already proven this by demonstrating the existence of extensions from the theorem that the Cantor set is totally disconnected. We need something to connect the points of the Cantor set within the continuum, to join them together, and these are the extensions. This is the reason why the Cantor set is not homeomorphic to the continuum. That is why the Cantor set, as a bare notion of the power set of ω is not a model of the continuum. Only a structure homeomorphic to the continuum can be its model. This homemorphic structure is provided by the derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, which becomes homeomorphic because it contains extensions in order to fill out the line and make it continuous and connected. Subsequently, I shall also demonstrate the existence of extensions by consideration of the Mahler classification of transcendental numbers [Section 16].

I shall also subsequently prove that there is continuum many extensions in the continuum [Section 17.4]. Since it takes $\mathfrak{c}$ many extensions to make up any interval of positive measure, it follows that the measure of an extension is also zero. However, I shall define measure of an extension to be **relatively zero**; precisely because any collection of continuum many of them has a positive measure. Subsequently, we shall see that the *S* numbers of the Mahler classification of transcendental numbers constitute extensions and we will understand why these numbers are relatively zero, because we will show them to have a substructure [Section 16.3.13].

To say that a set, *X*, has zero measure is to say that it may be placed inside an interval, *I*, whose measure may be made as small as we please. In symbols: -

$$\mu^*(X) = 0 \Leftrightarrow (\exists I)\big(I = (a,b) \subset \mathbb{R} \wedge X \subseteq I \wedge (\forall \varepsilon \in \mathbb{Q})\big(\mu^*(I) < \varepsilon\big)\big)$$

In order to avoid even the hint of circularity, we use $\mathbb{Q}$ in this definition rather than $\mathbb{R}$ ($(\forall \varepsilon \in \mathbb{Q})$; this is because that the continuum, $\mathbb{R}$, in our work is constructed from $\mathbb{Q}$, so it could be construed as circular to assume its existence in a definition of one of its members. We wish to define the arithmetical continuum just as Hardy did [Section 3.3] as "the aggregate of all real numbers, rational and irrational" and "suppose that the straight line … is composed of points corresponding to all the numbers of the arithmetical continuum, and of no others."

However, this is an interesting and moot point. Prima facie the need to use $\mathbb{Q}$ rather than $\mathbb{R}$ should not pose a problem, owing to the classical result that between any two rational numbers there is an irrational number. However, if it is possible to imagine that there is a point on the continuum that comes immediately after 0 in the interval, $[0,1]$, then this point could not be a rational number. So, then, there would be a real, $\zeta \in \mathbb{R}$ such that, for any extension point, *x*, the measure of *x* is greater than or equal to ζ; this is also what we mean by saying that measure of extensions is only relatively zero.

To say that a set, *X*, has zero measure is essentially to say that we have nothing in our number system with which to measure it. So in this sense, its measure is zero precisely because its measure is incommensurable with zero. If we had more than $\aleph_0$ many numbers in the skeleton, then it could become possible to compare such apparently zero measure sets with some other concept of measure. This is one of the ways in which models of the continuum may be constructed in which the Continuum Hypothesis is violated, and also a possible motivation for doing so.

10.5 The primitive notion of extension

In first-order set theory we have just one primitive notion, that of set membership. Nonetheless, the arithmetical continuum is a model of a physical reality – the continuum is always extended in space. We literally "see" this extension and that is how we know it, and it is part of our primitive, phenomenological concept of the continuum, from which we primarily abstract our scientific notions, whatever empirical confirmation we may subsequently give them. The question arises: can we derive the notion of extension from that of set membership? At first, it seems that we might do, for we can define an interval in any ordered set. However, here we realise that there are intervals in sets that do not comprise an extension, in the sense of space, because these sets are discrete and totally disconnected. A primitive notion of connectedness also goes with the concept of the continuum. Thus, from set-membership alone we cannot define or abstract the notion of extension. Extension is a second primitive notion required in order to form a model of the continuum. It is explicitly employed in the model displayed here when we imagine dividing an interval by means of a mere boundary; the interval being *a priori* extended. Then we divide the interval into that which is not extended and takes up no space whatsoever (a boundary), though it may yet be said to "exist", and that which remains extended, and yet has had something removed (an open interval).

It is often said that first-order set theory is sufficient to formalise every part of mathematics. In this monograph we clearly see that this is doubly false. Firstly, the notion of the continuum here is founded on the second-order Axiom of Completeness, which is universally allowed to be irreducible to first-order set theory. Secondly, we see that in addition to the primitive notion of set-membership we have a second primitive, that of extension.

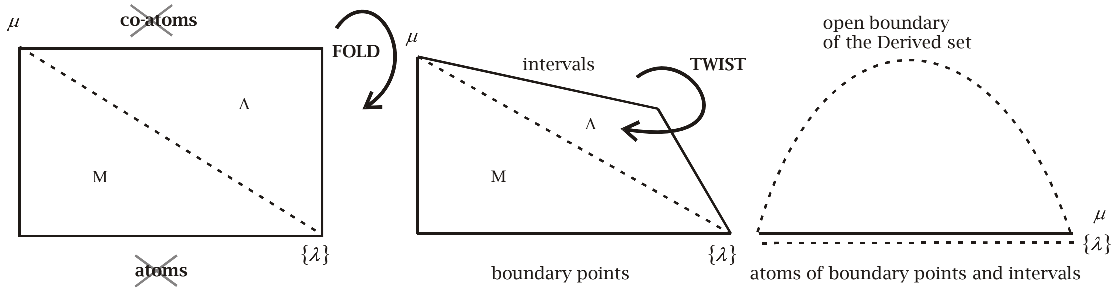
One often sees the claim that everything that can be expressed in a second-order theory can be translated into a many-sorted first-order theory. This claim is misleading. The properties of second-order logic distinguish it fundamentally from any first-order logic that is sufficiently strong to express Gödel's theorem. Second-order logic is categorical for arithmetic, first-order logic is not. For absolute clarity on this point see Boolos, George and Jeffrey, Richard [1980] , Chapter 18.

10.6 Modified image of the derived set

We began by assuming that the skeleton of the continuum was an actually infinite partition into $\aleph_0$ pieces. For the sake of exposition, there was an ambiguity in this, in that it was assumed that the partition could be into intervals, but the structure of those intervals was ambiguous. One way to consistently achieve this is to make the skeleton comprise of half-open intervals.

The null-megre decomposition of the continuum reveals that we have to go further, and decompose the skeleton into two parts – one comprising boundary points clustered on a rational number, the other comprising open intervals.

We are treating the intervals here as co-atoms, but they arise from the decomposition of the skeleton and may also be viewed as atoms. In this case we may draw the derived set as folded and twisted on itself.



We may demonstrate that the derived set is, subject to some caveats, a Klein bottle.[31]

11. Lowering the floor and notional atoms

11.1 More about Fin

We now seek a more concrete description of forcing by introducing a description within the Derived set in terms of lattice extensions, which is a process I shall call lowering the floor of the lattice.

The reason why $\mathbf{Fin} \cong \mathbf{2}^{<\omega}$ is atomless is because $\mathbb{N}$ is potentially, not actually, infinite. It would seem that singleton sets $\{1\} \leftrightarrow 1 \in \mathbb{N}, \{2\} \leftrightarrow 2 \in \mathbb{N}, \ldots$ are ideal candidates for atoms of $\mathrm{F}(\mathbb{N}) \cong \mathbf{2}^{<\omega}$, all the more so because the corresponding sets based on ordinals $\{1\} \leftrightarrow 1 \in \omega, \ldots$ are atoms of the Derived set of which **Fin** is a proper subset. But from within Fin, we find that it is always in a state of ongoing generation that has never been completed. [Sections 8.5 to 8.7]. It is a dense linear order. Suppose we take it as completed as some finite stage of its generation; then what we obtain is a <u>finite set of notional atoms</u>: $X_n = \{\{0\},\{1\},\ldots,\{n\}\}$ and the <u>finite</u> Boolean algebra $\mathbf{2}^n$. This is <u>atomic in itself</u> but as a representation of **Fin** always incomplete. No finite determination of <u>notional atoms</u> of **Fin** is ever sufficient to capture its potentially infinite and dense structure. A determination of notional atoms, $X_n = \{\{0\},\{1\},\ldots,\{n\}\}$, may potentially be enlarged in one of two equivalent ways: either (1) we add further atoms to the end of the list, $\{n+1\},\{n+2\},\ldots$, or we allow meets of the notional atoms to be defined: $\{0\} \wedge \{1\}, \{0\} \wedge \{2\}, \ldots$; the formation of these shows that the notional atoms never were true atoms in the first place; the two approaches are equivalent because the second of these would result in a re-labelling of the notional atoms to new notional atoms: $\{1\}^* = \{1\} \wedge \{2\},\ \{2\}^* = \{1\} \wedge \{3\}, \ldots$; Any finite determination of **Fin** is such that it may be embedded in another larger lattice with new notional atoms that subsume the previous ones. This process of replacing one determination of **Fin** by a larger one I call **lowering the floor** of the lattice. **Fin** is not a determinate Boolean algebra but rather a

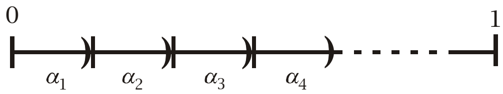
[31] This is the subject of a separate paper and part of my monograph of *Poincaré's thesis.*

potentially infinite collection of finite algebras: $\mathbf{2} \subset \mathbf{2}^2 \subset \mathbf{2}^4 \subset, \dots, \subset \mathbf{2}^{2^n} \subset \mathbf{2}^{2^{n+1}} \subset, \dots$. What we denote by $\mathrm{M} = \mathbf{P}(\mu)$ above [Section 8.3] is the supremum of this sequence. Because $\mathbb{N}$ is locally compact, so is **Fin**; likewise, we see that each member of the sequence above is compact (*a fortiori* locally compact), but **Fin** itself is not globally compact because it is not bounded above.

11.1.1 Model of the countable collection of intervals identified by their end-points

(Givant and Halmos [2009] p.25 et seq.)

Left half-closed intervals: $[a,b) = \{x \in X : a \le x < b\}$ define an *interval algebra*.



It is a countable collection of intervals identified by one end points. Each interval is an open subset of $[0,1]$ and hence atomless.

Fin is equivalent to this countable collection of intervals identified by their end-points. The intervals are connected and hence do not provide atoms. If either we strive to create totally disconnected sets by taking countable intersections of these intervals and so reach down to ultrafilters, or we strive to create larger intervals by taking unions of these intervals and so build up to a maximal ideal, neither operation can be completed. We are always left with sets that contain infinite collections of points in both directions. This is a model of the atomless algebra.

11.2 Generic sets

Generic sets were introduced by Cohen [1966] in order to prove (1) that there is a model of set theory (ZF) in which there are non-constructible sets, and (2) that the Axiom of Choice is independent of the axioms of ZF set theory. They are usually introduced in the context of a minimum transitive model of set theory. In this paper we are working directly in the Boolean valued model of a sufficient subset of the axioms of ZFC to define our primary object of enquiry, which is the derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, together with its non-atomic, countable subset, **Fin**. Our background logic is second order. Therefore, we shall drop references to the minimum model of ZFC, which is a requirement of first-order set theory. For us, first-order set theory is an important and incredibly powerful *language*, but we feel free to supplement it in any way necessary, particularly by second-order principles, and even by unformalised natural language argument where required.

11.3 Meets lie below the notional floor

We start with **Fin** with notional atoms $\{1\},\{2\},\{3\},...$; any lattice point (corresponding to a set) existing in **Fin** is represented by sets of numbers. For example $\{1\}$, $\{1,2\}$, $\{0,3,7\}$ are lattice points. Each lattice point defines a principal filter. For example, the filter $\text{filter}(1)$ corresponds to the filter which contains every set in which 1 appears: $\text{filter}(1) = \{\{1\},\{1,2\},\{1,3\}, \ldots ,\{1,2,3\},\{1,2,4\}, \ldots\}$. This can also be denoted, $1 \in x$.

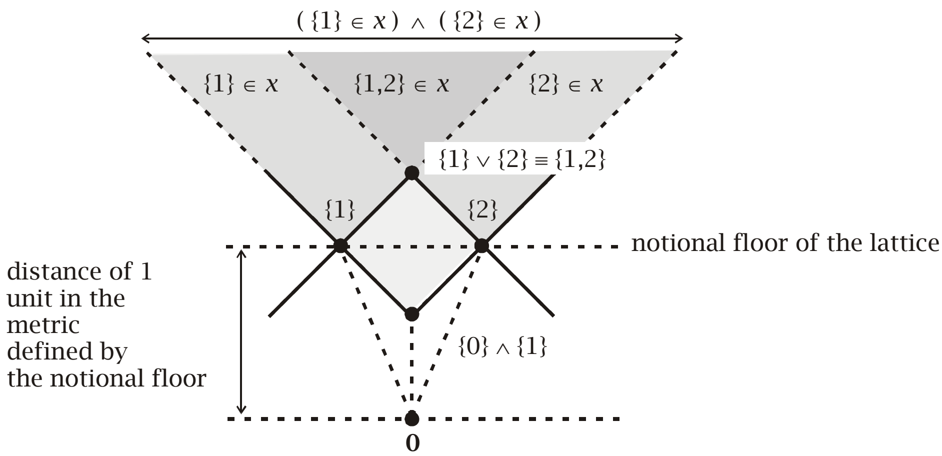
11.3.1 Example

$1 \in x$ corresponds to the filter which contains every set in which 1 appears: -

$1 \in x = \{\{1\},\{1,2\},\{1,3\}, \ldots ,\{1,2,3\},\{1,2,4\}, \ldots\}$

$\{1,2\} \in x = (1 \in x) \vee (2 \in x) = \{\{1,2\},\{1,2,3\},\{1,2,4\}, \ldots ,\{1,2,3,4\},\{1,2,4,5\}, \ldots\}$

In **Fin** there are no atoms; however, relative to a numbering of the lattice, there is a <u>notional floor</u>; this floor can be lowered, so that below any level there is another level. Although **Fin** is not atomic we imagine that there is some level in the lattice corresponding the singleton sets $\{1\}$, $\{2\}$, $\{3\}$... that correspond to <u>notional atoms</u>. Because the singleton sets $\{1\}$, $\{2\}$, $\{3\}$... do not represent true atoms, their meets define filters. Given a non-atomic lattice with singleton sets, we cannot represent their meets by other singleton sets, but only by expressions of the form $(1 \in x) \wedge (2 \in x)$ and so forth.



By conjunctions (meets) of filters we lower the floor of the lattice.

11.3.2 Example

We may replace the notional atom $\{1\}$, $\{2\}$ and $\{3\}$ by infinite, incomplete sets of atoms, say: -

$\{1\} \equiv \{a,b,c,d,...\}$ $\{2\} \equiv \{a,b,d,...\}$ $\{3\} \equiv \{a,c,d,...\}$

$\{1\} \wedge \{2\} \equiv \{a,b,d...\}$

$\{1\} \wedge \{2\} \wedge \{3\} \equiv \{a,d,...\}$

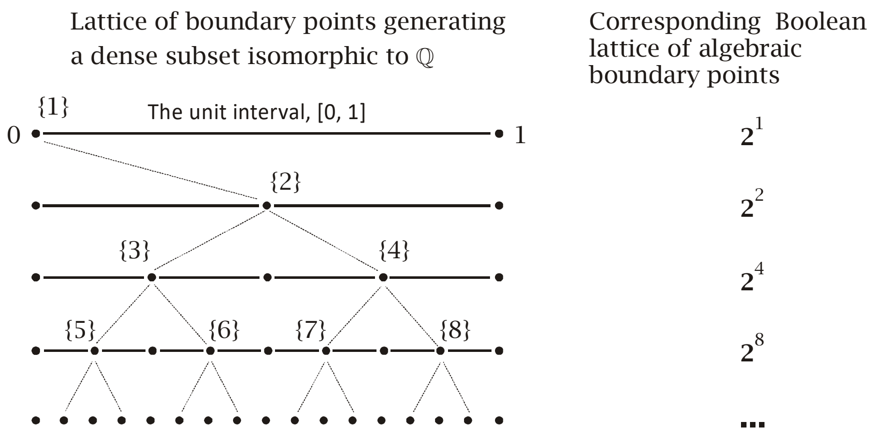
Alternatively, we can switch to an interval algebra.

12. Cohen forcing and Cohen reals

12.1 The tree and lattice of boundary points

We now assume that the skeleton, $\mathbb{N}_\lambda$, is decomposed into a sub-skeleton of atoms comprising a potentially infinite collection of boundary points, labelled $\{1\},\{2\},\{3\},...$, with the addition of a point at infinity, representing the neighbourhood of 1 in the unit interval $[0,1]$, and denoted $\{\lambda\}$, and a sub-skeleton of co-atoms, labelled $\omega-\{1\},\omega-\{2\},\omega-\{3\},...,\mu$, and representing proper intervals of the unit interval. The point at infinity must be an extension point, because it belongs to the filter of all cofinite subsets of ω, all of which are represent intervals.

Corresponding to the decomposition of the skeleton into meagre and not null sets, let us take the atoms to correspond to rational numbers that mark boundaries that divide the unit interval. We will recursively generate the segment of all algebraic numbers [Section 9.2] $\mathbb{N} \cong \{\{1\},\{2\},\{3\},...\}$ by introducing these rational numbers systematically into the unit interval by a method of bisection similar to the construction of the Cantor set. This generates a **lattice of boundary points.**



All the points of the skeleton of atoms are generated by this potentially infinite recursion. In the process, all algebraic numbers are also generated, since $\mathbb{N} = \mathbb{Z}^+$ is a basis for the ring of algebraic numbers, denoted $R[\mathbb{Z}^+]$ [Section 9.2]. The lattice generated is isomorphic to **Fin**. As already indicated any set of these points is both meagre and null.

Completion of the lattice requires that we take an actually infinite number of iterations of the construction; that is ω iterations as opposed to $<\omega$. We then obtain 2^{ω} branches each of length ω, each representing the binary expansion of a real number. Because these numbers do not belong to the potentially infinite part of the tree, they are not algebraic; hence they are transcendental numbers. Similarly, they are not members of any meagre set, and are hence Cohen reals.

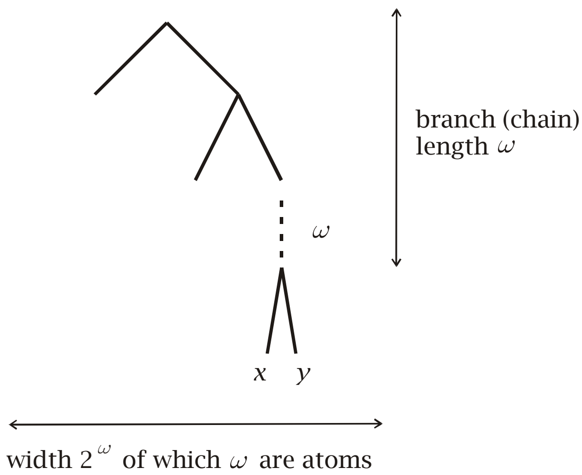
12.1.1 Definition, Cohen forcing

The process of extending a potentially infinite branch in this lattice of boundary points to an actually infinite one is called **Cohen forcing**.

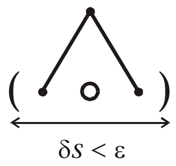
Cohen forcing completes **Fin** by adding a Cohen real not belonging to any meagre set in **Fin**. (In fact, **Fin** comprises only meagre sets.)

12.2 The character of a Cohen real

The Cantor tree comprises an infinite collection of infinite branches. Each branch arises from the branching of a node into a pair of nodes. When we take a path of length actually ω nodes there is the question as to whether or not this property is violated. Consider an actually infinite tree of boundary points: -



In fact, the question is partly a pseudo question. As the branches are of actually infinite length ω, the branches x and y in the above diagram, corresponding to binary expansions of a real number, share the same binary expansion. A Cohen real is defined by a branch length of ω; any branch longer than this is the same Cohen real. When the path is of length actually ω the two "last" branches are so close together that the separation is absolutely incommensurable with 0. In this way the two "last" branches become identified. We may regard the Cohen point a member of the "space", "interval" or "extension" between the "last " two branches.



The two terminal branches are identified because the distance (measure) between them is so small that it is less than any commensurable quantity. The Cohen real may be pictured as either an infinite branch in the lattice of boundary points, or its correspondent Boolean algebra, or as a point occupying the interval (or extension) between them. Cohen reals do not belong to any meagre set; they belong to the interior of open sets; the entire set of Cohen reals is an open dense subset of the continuum, $\mathbb{R}$. This makes Cohen reals into members of the A set above [Section 10.2] in the meagre-null decomposition of the line.

Members of meagre sets form the boundaries of other sets; a meagre set is nothing but a boundary without interior; meagre sets have empty interior.

12.3 Definition, Aronszajn tree

An ω_1-Aronszajn tree, T, is an ω_1-tree such that every chain in T is of cardinality ω.

12.4 Result (+)

The collection of boundary points is an ω_1-Aronszajn tree.

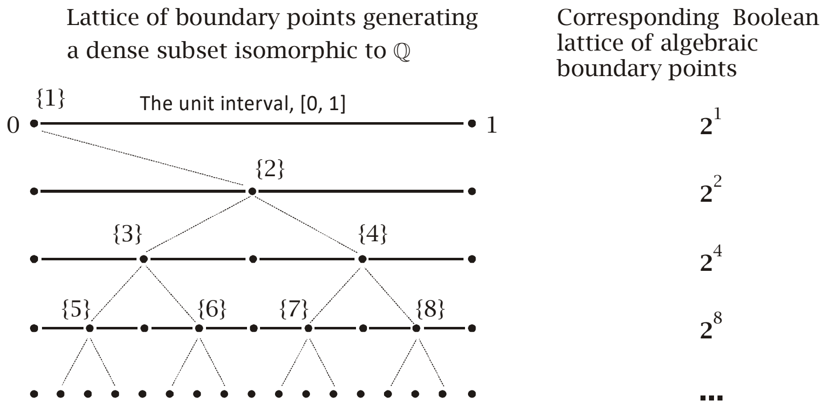
<u>Proof</u>

From the diagram we see that in the tree of boundary points the chains have length ω, whilst the tree as a whole has cardinality $\mathfrak{c} \geq \omega_1$.

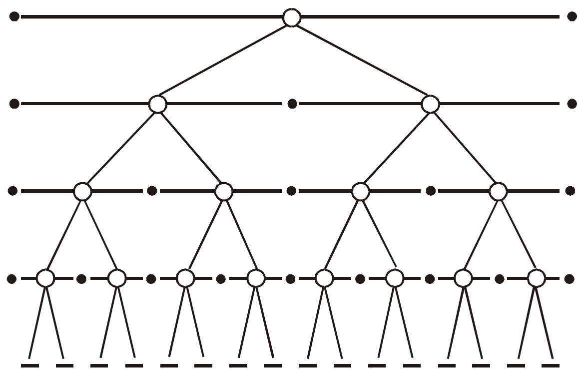
13. Amoeba reals

13.1 The tree of intervals

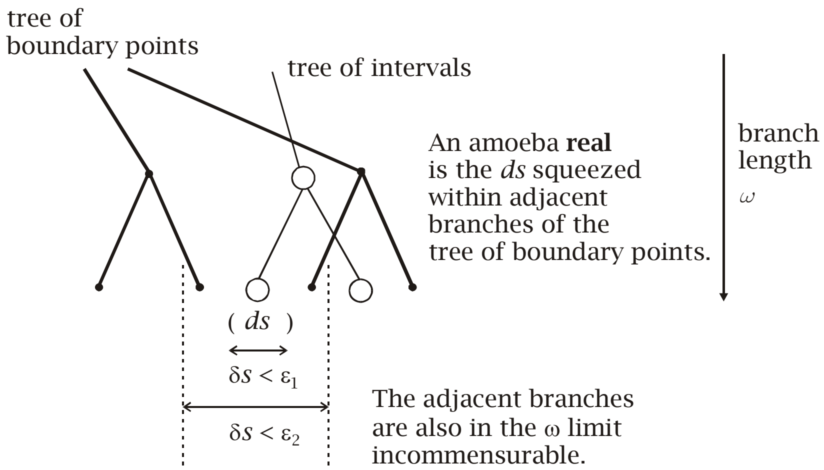
A real is a transcendental number that does not belong to any null set in the derived set of the one point compactication of the skeleton of the continuum. When we discussed Cohen reals, above, we started with the notion of removing boundary points from the unit interval and generating the atoms of the skeleton: -



By shifting our attention from the points, to the intervals that remain, we obtain a tree of intervals.



In this tree each node represents an open interval, each of which is not null (i.e. has positive measure). The boundaries are marked in black in the above diagram. At each level they comprise a null set. So the amoeba reals belong to the open intervals. At no finite level is any amoeba real generated. Only when the branches become actually infinite in length, ω, do we have amoeba reals. In this case the amoeba reals are these intervals, or rather extensions, as we have defined extensions above. The amoeba real may be identified with an extension, *ds*, such that $\mu(ds) < \varepsilon$ for all $\varepsilon \in \mathbb{Q}$. When the branches of intervals become actually infinite, ω, in length their length (or measure) is incommensurable with 0; it is the branch of the tree of open intervals itself that is the amoeba real, because the boundary points that separate these branches is collectively a null space, and amoeba reals do not belong to null sets.



The Boolean algebra generated by the boundaries (the black spots in the diagrams) make a Cantor set. This Cantor set is made up of a part of cardinality $\aleph_0$ corresponding to all the algebraic numbers in $[0,1]$, and the set of end points, of cardinality $\mathfrak{c} = 2^{\aleph_0}$ of Cohen reals. The whole structure is a perfect tree.

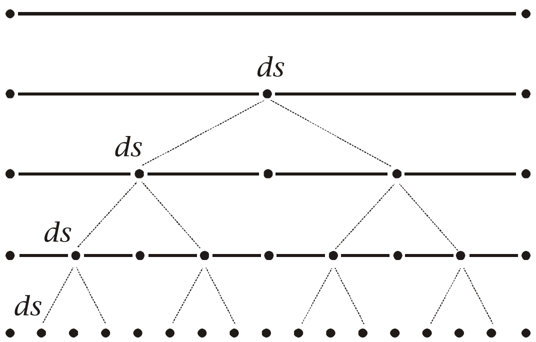
However, the collection of amoeba reals cannot be a Cantor set, that is to say, a perfect set of cardinality $\mathfrak{c} = 2^{\aleph_0}$. The reason is that it is a proper subset of $[0,1]$, so that if it were a perfect, Cantor set, it would then have to be a nowhere dense set of measure zero. But this collection, *R*, taken as a whole still has the same measure as the entire interval, since only zero measure boundaries have been removed from it. Consequently, it cannot be a perfect, Cantor set. Note, also, that the derived set of the amoeba reals must include all its boundaries; so the derived set is the entire continuum. The derived set of any perfect set is, by definition, itself.

We can see that the collection is non-cumulative. Looking on the points not as points but as gaps, the measure of the gaps is zero. There are no null sets in the pieces (which are intervals); there are only null sets of the gaps. Even if we take an infinite collection of the gaps such a collection cannot amount to a set of positive measure. At every finite stage no amoeba reals whatsoever have been introduced into the lattice of intervals. Every member of the potentially infinite lattice of intervals is an interval and has positive measure; i.e. is not null. In the limit any amoeba real must belong to the left over part. The line has been divided into 2^{ω} parts each of which has incommensurable measure, and yet the sum of which equals the measure of the whole line: $\mu(\mathbb{I}) = \mu[0,1] = 1$.

But let us observe that the cardinality of the set of amoeba reals is $\mathfrak{c} = 2^{\aleph_0}$. Although this set is not perfect, the above diagram does show that it is generated by a perfect set, one that contains amalgamations of these amoeba reals into open intervals of positive measure. So $\mathfrak{c} = 2^{\aleph_0}$ amoeba reals are generated.

13.2 Iteratively removing extensions

We need to construct a representation of the amoeba reals that clarifies the fact that entire set of amoeba reals is not a perfect set. Let us then imagine that we already have the set of all amoeba reals, which are extensions as we defined them above; let any amoeba real be designated *ds*. According to the following diagram we construct a recursive process of removing all these *ds* from the unit interval.



When the branches become actually infinite in length, ω, we shall have removed all the amoeba reals. In this diagram the other numbers, being the algebraic numbers and the Cohen reals, must belong to the pieces that were left over. In the limit, at the ω th stage, these have shrunk to intervals of length $\delta s < \varepsilon$, which are incommensurable with 0. This confirms that the Cohen reals are boundary points belonging to open intervals of zero measure.

Let us count the number of Cohen reals and amoeba reals that are constructed in this process.

stage or level	open intervals converging to Cohen reals	new random reals, *ds*, added	cumulative *ds*
0	$2^0 = 1$	0	0
1	$2^1 = 2$	$2^0 = 1$	$2^0 = 1$
2	$2^2 = 4$	$2^1 = 2$	$2^0 + 2^1 = 3$
3	$2^3 = 8$	$2^2 = 4$	$2^0 + 2^1 + 2^2 = 7$
...	...	...	...
ω	2^ω	$\left\|2^{<\omega}\right\| \leq \omega$	$\sum_{n=0}^{\infty} 2^n = 2^\omega - 1 \cong 2^\omega$
$\aleph_0$	$\mathfrak{c}$	$\aleph_0$	$\mathfrak{c}$

Thus there are as many boundary points as there are extensions, *ds*. However, we also claim that the collection of *ds* is a Suslin tree.

13.2.1 Definition, the countable chain condition

Let X be a topological space. Then X has the ccc iff there is no uncountable family of pairwise disjoint open subsets of X. Alternatively, if every antichain in X is at most countable.

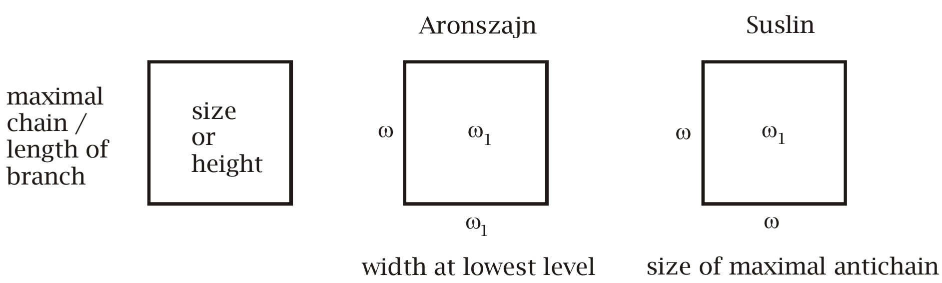
Remark

It is often said that this condition would be better described as the "countable antichain condition". This is true.

13.2.2 Definition, Suslin tree

A ω_1 - *Suslin tree* is a tree T such that $|T| = \omega_1$ in which every chain and antichain has cardinality $< \omega_1$. Another way of putting this is that a tree is Suslin if it has height or size ω_1, branches of length $\leq \omega$ and has the countable chain condition.

It is a result that all Suslin trees are Aronszajn but not conversely. The following is a diagrammatic interpretation of the difference between an Aronszajn and a Suslin tree.



Result (+)

The set of all amoeba reals is a Suslin tree

Proof

We need to generate cumulatively a set of $2^{\omega} \geq \omega_1$ amoeba reals. Examination of the diagram of the tree of amoeba reals shows that at each level the tree has the countable chain condition, since each *ds* is removed independently of the others. The tree has maximal chain of length ω. As the addition of new amoeba reals lags behind the number of open intervals by one level, at this stage we have added only ω linearly independent amoeba reals, *ds*. The cumulative total is $\sum_{n=0}^{\infty} 2^n = 2^{\omega} - 1 \cong 2^{\omega}$. Hence this is a Suslin tree.

There is a result in Bartoszynski and Judah [1995] (p.139) that Cohen reals produce a Suslin tree. That is right. It is when we remove boundaries and complete this process that we also create a Suslin tree. But the set of Cohen reals is not the Suslin tree, because it is a perfect set. Therefore, we must look elsewhere for the Suslin tree, and the only place to look for it is in the collection of amoeba reals; it is the set of amoeba reals that collectively comprises a Suslin tree. They observe, "… amoeba reals do not produce Suslin trees". This is also true. It is the Cohen forcing that produces the Suslin tree.

13.3 When is a perfect set not perfect?

13.3.1 A puzzle

We have another puzzle that needs clarification. The Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, is our interpretation of the Cantor set, $\mathbf{2}^{\omega}$, as a model of the continuum. The Cantor set is perfect but nowhere dense. The Derived set is perfect, but continuous. We have now shown that the Derived set has two proper subsets:

A A perfect set isomorphic to the Cantor set comprising all algebraic numbers and Cohen reals; also nowhere dense.

B A set of cardinality continuum that is dense comprising all amoeba reals.

13.3.2 Resolution of this puzzle

Once again we must eschew the notion that the Cantor set is an unambiguous structure. There is a minimum structure which corresponds to the complete binary tree: $\mathbf{2}^{\omega} = \{0,1\}^{\omega}$, which is also the set of all binary sequences: $\omega \to 2$. This has cardinality continuum, $\mathfrak{c} = 2^{\aleph_0}$. This is not a model of the continuum. As such it is ambiguous and under-determined. The Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, is a model of the continuum. Because $\operatorname{card}(\mathbb{N}_\lambda) = \omega = \aleph_0$ it has as many points as the Cantor set, and so is, in a manner of speaking, a type of Cantor set: for instance, we have, $\operatorname{card}(\mathbf{2}^{\mathbb{N}_\lambda}) = \operatorname{card}(2^{\omega})$. Within this structure we identify two disjoint subsets:

1. The set of all algebraic numbers and Cohen reals, of measure zero. A perfect set, isomorphic to the Cantor set, $\mathbf{2}^{\omega} = \{0,1\}^{\omega}$, and nowhere dense.
2. The set of all amoeba reals, of measure 1 in the unit interval. Not a perfect set. Of cardinality continuum and a Suslin line.

If we pair off the skeletons of the two sets, we regenerate the skeleton of the continuum: $\mathbb{N}_\lambda$, and so recover the Derived set as a model of the continuum. There is more than one type of Cantor set, and to make a continuum, we need two of them.

14. Forcing and generic sets

14.1 A language to describe the lattice

There are alternative approaches to the discussion of generic sets. Cohen's original paper (Cohen [1966]) allowed a distinction between a language describing partial orders and set theoretic models to which the language refers. In the approach of Kunen [1980] the language and the model are recombined, and Kunen is at pains to stress that all the partial orders in question already exist in the model. The aim of the language is to describe how models can be extended, but Kunen does this from within the ground model. Kunen's approach is forced upon him by the need to work entirely within first-order set theory. The idea of a language talking about another structure does not cohere readily with first-order set theory, so Cohen's original intuitive approach had to be modified.

In this paper the background logic is already second order, so we are free to adopt the more "natural" approach of Cohen.

So we will assume that there is a language *K*, sufficient to describe partial orders. Our ground model shall be $L_0 = \text{Fin} = \text{F}(\omega) \cong 2^{<\omega}$. Because this lattice is not atomic, it is possible to lower the floor and obtain lattice points below it. This is represented by an embedding of the lattice L_0 in a larger lattice L_1. Iteration of this process creates a sequence of lattice extensions $L_0, L_1, \dots, L_k, L_{k+1}, \dots$. We use the language *K* to discuss the relations between any lattice L_k in this sequence and its extension L_{k+1}. The Cantor set, 2^ω, is the limit of a sequence of lattice extensions $L_0, L_1, \dots, L_k, L_{k+1}, \dots$ starting with $L_0 = \text{Fin} = \text{F}(\omega) \cong 2^{<\omega}$; we may write, $2^\omega = L_\omega = \lim\limits_{n\to\infty}\{L_n\}$.

Cohen's original paper was inspired by Galois theory and the concept of algebraic and transcendental field extensions. There is a deep and in fact exact analogy between Galois theory and the theory of generic extensions.

We need a convention for distinguishing between statements in the language *K* and corresponding relations within a lattice to which it refers. The lattice L_0, has notional atoms $\{m\}$. To the lattice filter $\text{filter}(m) = m \in x$ there corresponds the statement in *K*, $\overline{m} \in \overline{x}$, where $\overline{m}$ and $\overline{x}$ are said to be *labels* or *names* of *m* and *x* respectively. The expression $\overline{m} \in \overline{x}$ is a *statement* of the language *K.* In a countably infinite lattice L_0 with notional singleton sets representing notional atoms, the singleton (atom) $\{1\}$ defines a filter of which $\{1\}$ is always a subset of every element.

14.1.2 Example

$1 \in x = \text{filter}(1) = \{\{1\}, \{1,2\}, \{1,3\}, \dots, \{1,2,3\}, \{1,2,4\}, \dots\}$

Denoting the elements of the filter by *x*, we have: -

$(\forall x)(x \in \text{filter}(1) \text{ iff } 1 \in x \text{ iff } \{1\} \subseteq x)$

The corresponding statements in the language are $\overline{1} \in \overline{x}$ or $\overline{x} \in \overline{\text{filter}(1)}$.

The language closely matches the lattice, but the language has other statements and symbols that enable it to <u>talk about</u> the lattice and so discuss its extensions and embeddings in larger lattices. We introduce the symbol $\Vdash$ to denote a relation called **forcing** between statements of K to describe relations between filters in L_0 and lattice extensions, $L_0, L_1, \dots, L_k, L_{k+1}, \dots$. Whenever the lattice point q is contained in the filter defined by the lattice point p and the two are connected by a finite or locally compact <u>proof path</u> [Section 5.7] this is denoted by $\overline{p} \vdash \overline{q}$.

14.1.3 Example

In a lattice L let $p = \{1\}$, $q = \{1,2\}$ then $p \vdash q$.

In the language we have $\overline{p} \equiv \overline{1} \in \overline{x}$, $\overline{q} \equiv \overline{\{1,2\}} \in \overline{x}$ and $\overline{p} \vdash \overline{q}$.

The relation of forcing is such that $\overline{p} \vdash \overline{q} \;\; \Rightarrow \;\; \overline{p} \Vdash \overline{q}$, but the forcing relation extends the idea of deductive consequence of statements $\overline{p}$ beyond that of deductive inference within the lattice and its finie or locally compact proof paths. The relation of forcing is thus connected to that of **logical consequence** in that it says more about the lattice than is captured in the relation of locally compact proof paths: $\overline{p} \vdash \overline{q}$. If a lattice is incomplete then we have some lattice points, p, q such that $\overline{p} \vDash \overline{q}$ but $\overline{p} \nvdash \overline{q}$. Nonetheless, in the sequence of lattice extensions $L_0, L_1, \dots, L_k, L_{k+1}, \dots$ none are incomplete in this sense; but the limiting case: $2^{\omega} = L_{\omega} = \lim_{n\to\infty}\{L_n\}$, which is a model of Gödel's theorem, is incomplete.

The distinction between forcing and consequence is as follows. When we consider consequence, where we have incompleteness, we have a prior conception of the lattice L that has non-globally compact proof paths. By contrast, the forcing language describes relations within a lattice L that is complete in this sense, but enables one to discuss the relation of that complete lattice to other lattices that are generic extensions of it. So forcing is a relation <u>between</u> lattices whereas consequence is a relation <u>within</u> a given lattice.

14.2 The construction of generic sets

We have a lattice L_0 and a language K that is equipped with a relation of forcing $\Vdash$ subject to the rule, $\overline{p} \vdash \overline{q} \Rightarrow \overline{p} \Vdash \overline{q}$ where $\overline{p} \vdash \overline{q}$ iff there is a locally compact proof path in L_0. The language K is countably infinite and hence all statements of K can be recursively enumerated; this corresponds to a recursive enumeration of the lattice points of L_0.

We now establish a series of rules for the generic construction which leads to a series of lattice extensions $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ and the definition of a *generic set.* Let $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$ be any recursive enumeration of the statements of *K*. These are statements of the form $\overline{m} \in \overline{\text{filter}(\zeta)}$ where *m* is a lattice point and $\overline{\text{filter}(\zeta)}$ of L_0.

14.2.1

Rule 0

We start with a consistent collection $\overline{p_0}$ of statements.

Notation

We denote a conjunction of statements by $\overline{p_k}$. This corresponds to: -

1. To a filter denoted $\text{filter}(p_k)$.
2. If the filter is principal, then to a lattice point, which is the minimum point of the filter, and denoted $\{p_k\}$.

At all finite stages of construction, the filter is principal, but at the limiting stage after ω iterations, it is not principal. In the language, *K*, we have $\overline{q} = \lim_{k \to \omega} \overline{p_k}$, as a relationship between statements. At the limiting stage this defines a **generic ultrafilter**, $U = \text{filter}(q)$, which we can prove to exist in the derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$, but there is no lattice point, $\{q\}$, lying in $\mathbf{2}^{\mathbb{N}_\lambda}$. If it were an element of $\mathbf{2}^{\mathbb{N}_\lambda}$, then $\{q\}$ would be one of its atoms. Then $\mathbf{2}^{\mathbb{N}_\lambda}$ would have more than ω atoms. Suppose we partition $\mathbb{R}$ with a skeleton of $2^\omega > \omega$ atoms, then $\{q\}$ is an atom of the lattice $\mathbf{2}^{\mathbb{R}} \cong \mathbf{2}^{2^\omega}$. So we may regard $q \in \mathbb{R}$ as a real number, and $U = \text{filter}(q)$ as its **real number generator**. It is this real number generator, or sequence, that lies in the Cantor set, not the real number itself.

The statements, $\overline{p_k}$, are also referred to as **conditions**, because they convey information about members of a filter. The collection $\overline{p_0}$ represents <u>finite information</u> about a generic set $\text{filter}(q)$, and comprises a finite list of <u>conjunctions</u> of the form $\overline{m} \in \overline{q}$ or $\overline{m} \notin \overline{q}$. For example, $\overline{p_0} = \left(\overline{3} \in \overline{q}, \overline{47} \notin \overline{q}, \overline{932} \in \overline{q}\right)$. Here the round brackets is a device of my own to emphasise that this is not a disjunctive set, but a conjunctive one. But it the literature it is customary use curly brackets as well; this is presumably so that we can use the language of subsets to describe relations between conditions, which then form a partial order, denoted, $\mathbb{P}$.

14.2.2 Example

$\overline{p_0} = \{\overline{3} \in \overline{q},\ \overline{47} \notin \overline{q},\ \overline{932} \in \overline{q}\}$

$\overline{p_1} = \{\overline{3} \in \overline{q},\ \overline{47} \notin \overline{q},\ \overline{932} \in \overline{q}, \overline{81} \in \overline{q}\}$

$\overline{p_0} \subset \overline{p_1}$

Since $\overline{p_0}$ describes a lattice meet and not a join, we could also write: -

$\overline{p_0} = (\overline{3} \in \overline{q}) \wedge (\overline{47} \notin \overline{q}) \wedge (\overline{932} \in \overline{q})$.

This shows that $\overline{p_0}$ does not correspond to a lattice point such as $\{3, \neg 47, 932\}$, which lies above $\{3\}$ $\{47\}'$ and $\{932\}$. The lattice point $\underline{p_0}$ <u>would lie below</u> the lattice points $\{3\}$ $\{47\}'$ and $\{932\}$ in the lattice <u>if the lattice permitted it</u>, and not above them. Since $\{3\}, \omega - \{47\}, \{932\}$ are already singleton or co-singleton sets, that is, notional atoms or co-atoms, the meet $\underline{p_0} \equiv \{3\} \wedge \{47\}' \wedge \{932\}$ does not lie in L_0; however, L_0 is embedded in a larger lattice in the recursive sequence, $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ obtained by recursively lowering the floor [Section 11]. Generic sets are constructed by rules for systematically generating a recursive sequence of lattice extensions, which is also a systematic process of lowering the floor. The generic set itself is an ultrafilter constructed as the limit of this recursive process. It is not itself recursive, since the recursion can only generate successive members of the sequence of lattice extensions, and not the limit.

Now we have to provide an example of a set of rules for generating a sequence of statements $\{\overline{p_k}\}$ that act as successive approximations to a statement $\overline{q} = \lim\limits_{k\to\infty} \overline{p_k}$ that defines a generic filter *q*.

14.2.3

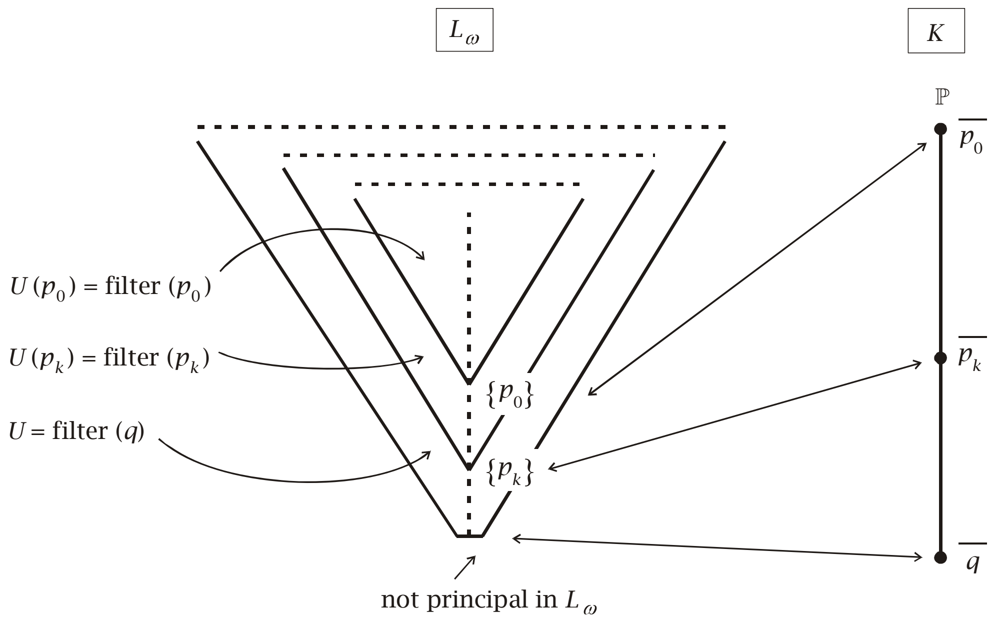
Rule 1

If $\overline{S_k} \in \overline{p_{k-1}}$ then $\overline{p_k} = \overline{p_{k-1}}$. Since $\overline{q} = \lim\limits_{k\to\infty} \overline{p_k}$ this entails $\overline{p_k} \subseteq \overline{q}$.

<u>Example</u>

Let $\overline{p_0} = \{\overline{3} \in \overline{q},\ \overline{47} \notin \overline{q},\ \overline{932} \in \overline{q}\}$. Suppose $\overline{S_1} \equiv \overline{3} \in \overline{q}$ then $\overline{p_1} = \overline{p_0}$.

This rule serves to ensure that p_k lies in $U = \text{filter}(q)$. We have: $\underline{q} \le \underline{p}$ iff $\overline{p} \subseteq \overline{q}$; $\underline{q} \le \underline{p}$ describes lattice points of the sequence $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$, whereas $\overline{p} \subseteq \overline{q}$ is in the language *K*. Notice that the relation is inverted.



On the left we have a supraset relation, $p_0 ... \supseteq p_k \supseteq p_k \supseteq ...$ <u>in the sequence of lattice extensions</u> whereas on the right we have a subset relation $\overline{p_0} \subseteq ... \subseteq \overline{p_k} \subseteq \overline{p_{k+1}} \subseteq ...$ <u>in the language</u>. This subset relation also makes the conditions in K into a partial order, which we denote $\mathbb{P}$: $p \leq_{\mathbb{P}} q$ iff $p \subseteq q$. We will subsequently adopt a more generalised and abstract description of forcing, in which only partial orders will be considered. Partial orders with differing properties define different forcings, and consequently different generic ultrafilters.

14.2.4

Rule 2

If $\overline{S_k} = \neg\overline{S_j}$ where $\overline{S_j} \in \overline{p_{k-1}}$, then

1. $\overline{p_k} \vdash S_j$ (That is, $\overline{p_k} \vdash \neg\overline{S_k}$.)
2. $\overline{p_k} \Vdash \neg\overline{S_k}$ and $\overline{S_k} \notin \overline{p_k}$. This also means $\overline{S_k} \notin \overline{q}$.

This rule prevents $\overline{p_k}$ and $\overline{q}$ from being inconsistent sets of conditions. This in turn means that the lattice points to which they correspond cannot be names of the **0** of the lattice; that is, neither $\underline{p_k} \neq \mathbf{0}$ nor $\underline{q} \neq \mathbf{0}$. This means that $\text{filter}(q)$, which is a generic ultrafilter, lies in neighbourhood of **0** (the zero of the lattice) but distinct from it.

At any finite stage the set $\overline{p_k}$ is finite (we say "contains finite information") so there are statements $\overline{S}$ of the form $\overline{m} \in \overline{q}$ or $\overline{m} \notin \overline{q}$ that are undecided at this stage. So we need a rule for deciding these statements as they come up in the recursive sequence of statements.

14.2.5

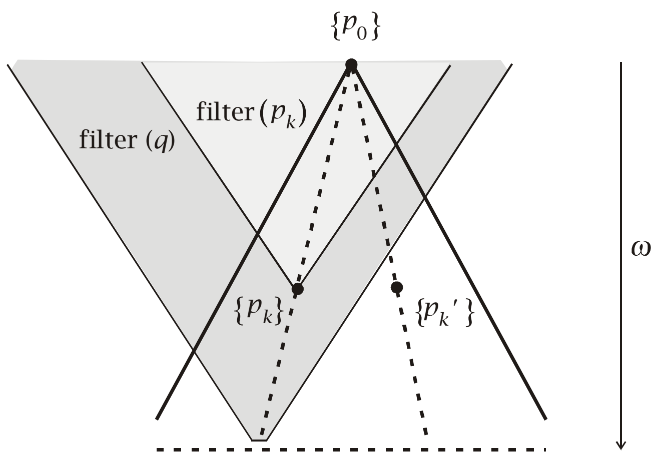
Rule 3

If $\neg\overline{S_k} \notin \overline{p_{k-1}}$ then at the stage k – 1 $\overline{S_k}$ has not yet been decided. (This means, $\overline{p_{k-1}} \nVdash \overline{S_k}$ and $\overline{p_{k-1}} \nVdash \neg\overline{S_k}$.) Then $\overline{p_k} = \overline{p_{k-1}} \cup \overline{S_k}$. This entails $\overline{p_k} \Vdash \overline{S_k}$ and $\overline{q} \Vdash \overline{S_k}$

This simply says that if at stage k – 1 we have $\overline{S_k}$ undecided, then we decide $\overline{S_k}$ at the stage k by adding it to $\text{filter}(p_k)$. This rule forces us to decide every statement whatsoever in the recursive sequence $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$.

14.2.6 Theorem, generic sets cannot be constructed by finite information

Together these three rules applied to a lattice point $\{p_0\}$ create an ideal at $\{p_0\}$. The generic set $\text{filter}(q)$ lies at on a globally non-compact path of actually ω lattice points distant from $\underline{p_0}$. There are many such paths winding through the ideal at $\underline{p_0}$, and the recursive sequence $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$ picks out one of these and homes in upon the generic ultrafilter, $\text{filter}(q)$. Starting at $\{p_0\}$ there are more than ω such paths winding through the idea; the number of paths is an ordinal β such that $\aleph_0 < |\beta| \le 2^{\aleph_0}$.



The supposition that the generic set $U = \text{filter}(q)$ can be constructed at any finite stage leads to a contradiction. Suppose $\overline{q} = \overline{p_k}$. Then since $\overline{p_k}$ is finite there must be some $n \in \mathbb{N}$ such that both $\{n\} \notin \text{filter}(q)$ and for the statement $\overline{S} \equiv (\overline{n} \notin \overline{q})$ we have $(\overline{n} \notin \overline{q}) \notin \overline{p_k}$. That is $\neg(\overline{n} \in \overline{q}) \notin \overline{p_k}$ and $\overline{S} \equiv (\overline{n} \notin \overline{q})$ is undecided at $\overline{p_k}$. Then by Rule 3 above we have $\overline{q} \Vdash (\overline{n} \in \overline{q})$, which is $(\overline{n} \in \overline{q}) \in \overline{q}$; hence $\{n\} \in \text{filter}(q)$. So we obtain

$\{n\} \in \text{filter}(q)$ and $\{n\} \notin \text{filter}(q)$, a contradiction. Hence, the supposition $\overline{q} = \overline{p_k}$, that q is constructed at some finite stage is false. This is because $\text{filter}(q)$ is defined by an <u>inductive procedure</u> that take us out of the sequences of lattices, $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$, of countably infinite lattice points to <u>transcend</u> it. Hence, no generic set is "constructible" <u>in the sense of recursively enumerable</u>. The generic sequence $\overline{q} = \lim_{k \to \infty} \overline{p_k}$ represents an inductive rule in the language, *K*, for enumerating a series of lattice points indefinitely below any given lattice point $\{p_0\}$, and the contradiction arises from assuming that $\{q\} = \{p_n\}$ can be completed at some definite point, which is the same as assuming that a generic set $\text{filter}(q) = \text{filter}(p_n)$ is completely defined by a lattice point $\{p_n\}$, which would make it a principal filter. The "non-constructibility" of a generic set arises from the fact that no countably infinite lattice is atomic, which entails that there are always lattice points lying below any notional atom, and the contradiction arises from assuming that a finite set of conditions $\overline{p_k}$ defines an atom. Only an atom could define $\{q\} = \{p_\omega\}$ and no such atom can be reached by any recursive procedure.

14.3 Numbers and functions

Transcendental numbers may be viewed as *both* numbers and functions. For example π is a number, but it is also a function: $\pi : \omega \to \omega, \pi \in \omega^\omega$ such that

$\pi(1) = 3$ $\quad\quad$ $\pi(2) = 1$ $\quad\quad$ $\pi(3) = 4$ $\quad\quad$ $\pi(4) = 2$

Every transcendental number encodes a function and every function codes a transcendental number. The ultrafilters generated by forcings comprise functions that generate transcendental numbers. It is the ultrafilters that belong to the derived set, not the transcendental numbers. Nonetheless, the derived set may be embedded in a larger lattice in which for each transcendental number, ξ, there is an atom, $\{\xi\}$.

15. Transcendental numbers

15.1 Liouville numbers

We have already seen that a real or complex number is algebraic if it is the zero of a polynomial with integer coefficients. It is a result that every algebraic number α is the zero of some irreducible polynomial *f* that is unique up to constant multiple. The degree of α, denoted $\partial\alpha$ is the degree of the polynomial *f*.

15.1.1 Liouville's theorem

Let α be an algebraic number with degree $n > 1$. Then there exists a $c = c(\alpha)$ such that $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$ for all $p,\ q \in \mathbb{N} > 0$.

<u>Proof</u>

Let $f(x)$ be an irreducible polynomial with root α. The mean value theorem is $f'(\xi) = \frac{f(b) - f(a)}{b - a}$, $a \le \xi \le b$. On substituting $\frac{r}{s} = b$ we obtain, $f'(\xi) = \frac{f(\alpha) - f\left(\frac{r}{s}\right)}{\alpha - \frac{r}{s}}$, $\alpha \le \xi \le \frac{r}{s}$. Hence, since $f(\alpha) = 0$, $. -f\left(\frac{r}{s}\right) = f'(\xi)\left(\alpha - \frac{r}{s}\right)$

If $\left|\alpha - \frac{r}{s}\right| \ge 1$ the result is trivially true because $\frac{1}{s^n} \le 1 \le \left|\alpha - \frac{r}{s}\right|$. Then, for the non-trivial case, suppose $\left|\alpha - \frac{p}{q}\right| < 1$. Then $\xi \le \frac{p}{q} - \alpha$, $|\xi| \le \left|\alpha - \frac{p}{q}\right| - 1$ and $|\xi| < 1 + |\alpha|$. As $|\xi| < 1 + |\alpha|$ we have ξ is close to α. At α we have $f'(\alpha) = 0$, so $f'(\xi) \to 0$ as $\xi \to \alpha$. That means that $|f'(\xi)| \to 0$, which means that for given ξ there is a $c = c(\alpha) > 0$ such that $|f'(\xi)| < \frac{1}{c}$. Thus $|f'(\xi)| < \frac{1}{c}$ where $c = c(\alpha) > 0$. From $-f\left(\frac{r}{s}\right) = f'(\xi)\left(\alpha - \frac{r}{s}\right)$ we obtain, $\left|f\left(\frac{r}{s}\right)\right| = |f'(\xi)|\left|\left(\alpha - \frac{r}{s}\right)\right|$. Then substituting $|f'(\xi)| < \frac{1}{c}$ gives, $\left|f\left(\frac{r}{s}\right)\right| = \frac{1}{c}\left(\left|\alpha - \frac{r}{s}\right|\right)$. This gives, $\left|\alpha - \frac{r}{s}\right| < c\left|f\left(\frac{r}{s}\right)\right|$. But f is irreducible, hence $f\left(\frac{r}{s}\right) \ne 0$ and the integer $\left|q^n f\left(\frac{r}{s}\right)\right| \ge 1$. Hence, $\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^n}$ as required.[32]

15.1.2 Example of a Liouville number

Let $\xi = \sum_{n=1}^{\infty} 10^{-n!} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^{3!}} + \frac{1}{10^{4!}} + \ldots = 0.110001000000000000000001000\ldots$.

Let

$$r_j = 10^{j!}\sum_{n=1}^{j} 10^{-n!} \qquad s_j = 10^{j!} \qquad (j = 1,2,3,\ldots)$$

Then $r_j,\ s_j$ are relatively prime rational integers.

[32] Barker [1975] states that an explicit value for *c* is given by $\frac{1}{c} = n^2\left(1 + |\alpha|\right)^{n-1} h$ where *h* denotes the height of α, which is the maximum of the absolute values of the coefficients of *f*.[32]

$$\left|\xi - \frac{r_j}{s_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < 10^{-(j+1)!}\left(1 + 10^{-1} + 10^{-2} + \ldots\right) = \frac{10}{9}\cdot\frac{1}{10^{(j+1)!}} = \frac{10}{9}\cdot\frac{1}{10^{(j+1)j!}} = \frac{1}{9}\cdot\frac{1}{10^{j\cdot j!}} < \frac{1}{\left(s_j\right)^j}$$

By Liouville's theorem, if ξ is algebraic, then $\xi > \frac{1}{\left(s_j\right)^j}$, so ξ must be transcendental.

15.1.3 Theorem, transcendental numbers are generic (+)

Where ξ is the transcendental number of the preceding example, $U_\xi = \text{filter}(\xi)$ is an ultrafilter and generic set in the derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$.

<u>Proof</u>

Let $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$ be a recursive enumeration of statements such that $\overline{S_n} \equiv \overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \overline{q}$.

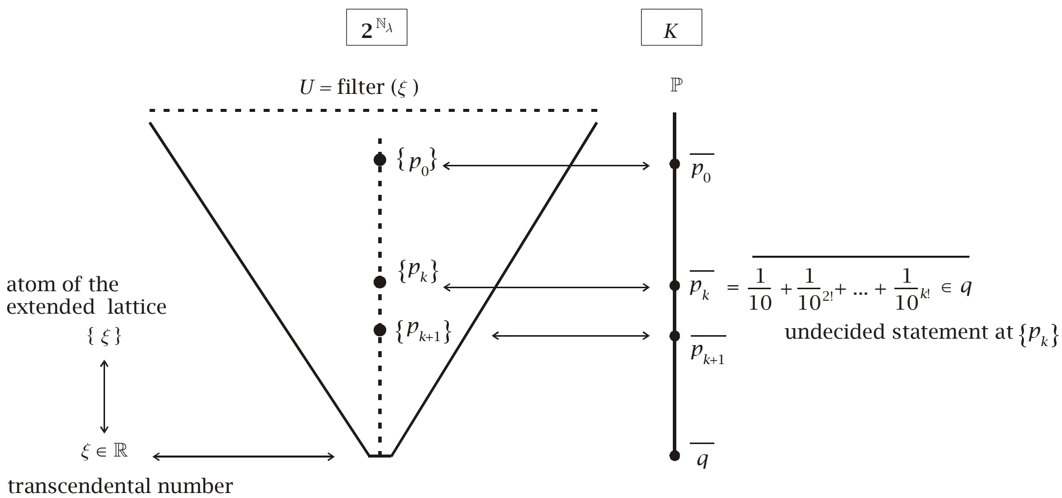
Let

$\{p_0\} = \{0\}$ $\qquad$ $\overline{p_0} \equiv \left\{\overline{\{0\} \in q}\right\} \equiv \left\{\overline{0} \in \overline{q}\right\}$

$\{p_1\} = \left\{\frac{1}{10}\right\}$ $\qquad$ $\overline{p_1} \equiv \overline{p_0} \cup \left\{\overline{\frac{1}{10}} \in \overline{q}\right\}$

$\{p_{k+1}\} = \left\{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}\right\}$ $\qquad$ $\overline{p_{k+1}} \equiv \overline{p_k} \cup \left\{\overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \overline{q}\right\}$

That is $\overline{p_{k+1}} \equiv \overline{p_k} \cup \overline{S_k}$. The following diagram illustrates this proof.



Let an ideal be defined by the inductive rule: -

$$\overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \overline{q} \Rightarrow \overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{(k+1)!}}} \in \overline{q}$$

Then, for a contradiction, let $\overline{q} = \overline{p_k}$ where $k \in \mathbb{N}$. This entails that ξ is algebraic. Then by Liouville's theorem $\xi > \frac{1}{(s_k)^k}$ where $s_k = 10^{k!}$. But by the preceding result $\xi < \frac{1}{(s_k)^k}$. Hence $\overline{q} \neq \overline{p_k}$ and $\text{filter}(q)$ is a generic set. Define $\xi \in \mathbb{R}$ such that $\text{filter}(q)$ is principal and $q = \{\xi\}$.

15.1.4 Definition, Liouville number

A *Liouville number* is any real number ξ that possesses a sequence of distinct rational approximations $\frac{p_n}{q_n}\,(n = 1,2,3,\ldots)$ such that $\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\omega_n}}$ where $\lim(\sup \omega_n) = \infty$.

In the section on the Mahler classification of transcendental numbers [Section 16] I will show all Liouville numbers are transcendental. Let us proceed to provide a second concrete example demonstrating that transcendental numbers are added to the continuum by generic ultrafilters. In Liouville's theorem the condition $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$ is a forcing condition on a generic sequence.

15.2 The number *e*

Hermite proved in 1873 that the number *e* is transcendental. The presentation of his theorem that follows here is due to Baker [1975].[33]

15.2.1 Hermite's theorem

e is transcendental.

<u>Lemma 1</u>

Let $f(x)$ be any polynomial of real coefficients of degree *m*. Let

$$I(t) = \int_0^t e^{t-u} f(u)\,du$$

where *t* is an arbitrary complex number and the integral is taken over the line joining 0 to *t*. Integration by parts gives: -

[33] Baker's treatment could be said to be "light". A fuller treatment is in Burger and Tubbs [2004], though that is not more perspicuous. For the purposes here it is not required to clarify every inference, since the aim here is soley is to show that Hermite's proof constructs *e* as a generic set.

$$\begin{aligned} I(t) &= \int_0^t e^{t-u} f(u)\,du \\ &= \left[f(u)\cdot -e^{t-u}\right]_0^t + \int_0^t f^{(1)} e^{t-u} \\ &= -f(t) + e^t \cdot f(0) + \int_0^t f^{(1)} e^{t-u} \end{aligned}$$

and repeated integration by parts gives: -

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t)$$

<u>Lemma 2</u>

Let $\overline{f}(x)$ denote the polynomial that is obtained from *f* by replacing each coefficient in *f* with its absolute value. Then

$$|I(t)| \le \int_0^t \left|e^{t-u} f(u)\right| du \le |t| e^{|t|} \overline{f}(|t|)$$

<u>Proof of the theorem</u>

1. Suppose *e* is algebraic. Then there exist integers $q_0, q_1, \ldots, q_n,\ n > 0$ such that

   $q_0 + q_1 e + \ldots + q_n e^n = 0$.

   Let

   $J = q_0 I(0) + q_1 I(1) + \ldots + q_n I(n)$

   where $I(t)$ is defined as in the lemma and $f(z) = z^{p-1}(z-1)^p \ldots (z-n)^p$ and p is a large prime. Substituting $I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t)$ we obtain: -

$$\begin{aligned} I(t) &= q_0 \left\{ \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) \right\} + q_1 \left\{ e\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(1) \right\} + \ldots + q_n \left\{ e^n \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(n) \right\} \\ &= q_0 \left\{ -\sum_{j=0}^{m} f^{(j)}(t) \right\} + q_1 \left\{ -\sum_{j=0}^{m} f^{(j)}(1) \right\} + \ldots + q_n \left\{ -\sum_{j=0}^{m} f^{(j)}(n) \right\} \\ &= -\sum_{j=0}^{m} \sum_{j=0}^{n} q_k f^{(j)}(k) \end{aligned}$$

   where $m = (n+1)p - 1$. We have $f^{(j)}(k) = 0$ if $j < p$ and $k > 0$ or if $j < p-1$ and $k = 0$. Hence, for all *j, k* except $j = p-1$ and $k = 0$ we may say that $f^{(j)}(k)$ is an integer divisible by $p!$. Furthermore,

   $f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p$

   from which it follows that if $p > n$, we have $f^{(p-1)}(0)$ is an integer divisible by $(p-1)!$ but not by $p!$ Then, if $p > |q_0|$, we have J is a non-zero integer divisible by $(p-1)!$. hence $|J| \ge (p-1)!$

This establishes one estimate for $|J|$ which is based on the first lemma and applies on the assumption that e is algebraic.

2. To obtain another estimate, note that $\bar{f}(k) \le (2n)^m$ and this combined with

$|I(t)| \le \int_0^t |e^{t-u} f(u)| du \le |t| e^{|t|} \bar{f}(|t|)$

from the second lemma, gives

$|J| \le |q_1| + \ldots + |q_n| n e^m \bar{f}(n) \le c^p$

where c is independent of p.

3. If p is sufficiently large the two estimates are contradictory, whence e cannot be a root of an irreducible algebraic polynomial.

15.2.2 Theorem *e* corresponds to a generic ultrafilter (+)

$U = \text{filter}(e)$ is an ultrafilter and generic set in the derived set.

Proof

Let $U = \text{filter}(e)$ be the consequences of everything that follows from the assumption that e is the zero of a function *f*; that is $f(e) = 0$. That is $\bar{\phi} \in \text{filter}(e)$ iff $\overline{f(e)} = \bar{0} \vDash \bar{\phi}$.

In the proof of Hermite's theorem we assume that e is algebraic. This is equivalent to the assumption that f is an algebraic function; hence there exist integers $q_0, q_1, \ldots, q_n, n > 0$ such that $q_0 + q_1 e + \ldots + q_n e^n = 0$. This information is encoded in a finite set $\overline{p_k}$, and the assumption is that $\bar{q} = \overline{p_k}$. This assumption is shown in Hermite's proof to decide two contradictory statements. Given

$J = q_0 I(0) + q_1 I(1) + \ldots + q_n I(n)$

where $I(t) = \int_0^t e^{t-u} f(u) du$, we have: -

1. $|J| \ge (p-1)!$
2. $|J| \le c^p$.

Since this is a contradiction violating the Fundamental Principle of Number Theory, these statements cannot be so decided. Hence $\bar{q} \ne \overline{p_k}$ and $U = \text{filter}(e)$ is a generic set.

The assumption that e is algebraic is equivalent to the claim that $\{e\} = \{p_k\}$ can be constructed at a finite ordinal level $k \in \mathbb{N}$ corresponding to a condition $\underline{p_k}$; but this assumption always leads to a contradictory statement at that level.

15.3 The Axiom of Constructibility

A **constructible set** is one that can be defined by a formula of first order set theory by an inductive procedure from the null set. The class of all constructible sets is denoted *L*. The universe of all sets is denoted *V*. The **Axiom of Constructibility** is $V = L$. The generic set construction was used by Cohen to establish that any model in which there is a generic set is one on which the Axiom of Constructibility is false. This is phrased in a conditional form: if there is a generic set, then the Axiom of Constructibility is false. Here we have shown that, subject to the Axiom of Completeness, there is a generic set. We have two specific examples, the ultrafilters defining a particular Liouville number and the number *e*. Hence we have proven the following theorem.

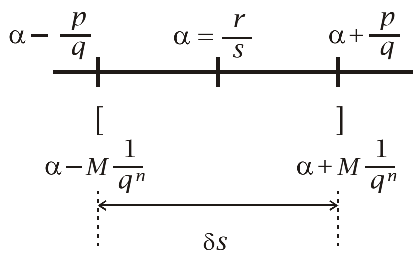
15.3.1 Theorem, the Axiom of Constructibility is false on the arithmetic continuum (+)

The Axiom of Completeness entails $V \neq L$.

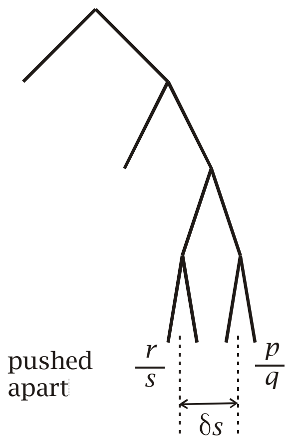
16. Mahler's classification of the transcendental numbers

16.1 Approximating one rational number by another

Let $\frac{p}{q}$ be a rational approximation of rational number $\frac{r}{s}$. We are always bounded away from $\frac{r}{s}$ by a constant multiple of $\frac{1}{q}$. In particular, $\left|\frac{r}{s} - \frac{p}{q}\right| = \frac{|rq - ps|}{sq} \geq \frac{1}{sq}$. We see that $\frac{p}{q}$ from any other rational approximation to it by a closed interval.



Burger and Tubbs comment that "Rational numbers are precisely those numbers that cannot be well-approximated by other rationals." In the potentially infinite Cantor tree, $2^{<\omega}$, rational numbers correspond to branches of finite length. These branches are separated by a finite interval; one cannot make the branch representing $\frac{p}{q}$ converge on the branch representing $\frac{r}{s}$ where $\frac{r}{s} \neq \frac{p}{q}$. It is for this reason that the set $\mathbb{Q}$ is totally disconnected in $\mathbb{R}$.

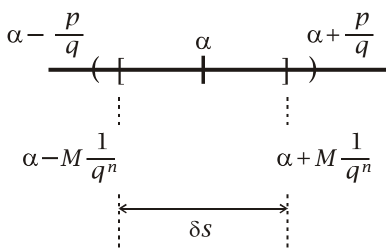


16.2 Irrational algebraic numbers

Irrational numbers may be subdivided into those that are algebraic, and those that are transcendental. The algebraic numbers behave in a way that is similar to the rational numbers. Liouville's theorem gives us a criterion for when a number α is algebraic.

16.2.1 Criterion for algebraic numbers

There exists a $c = c(\alpha)$ such that $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$ for all rational numbers, $\frac{p}{q} \in \mathbb{Q}$. It is customary to write $M = \frac{1}{c}$ and express this condition as, $\left|\alpha - \frac{p}{q}\right| > \frac{1}{Mq^n}$.

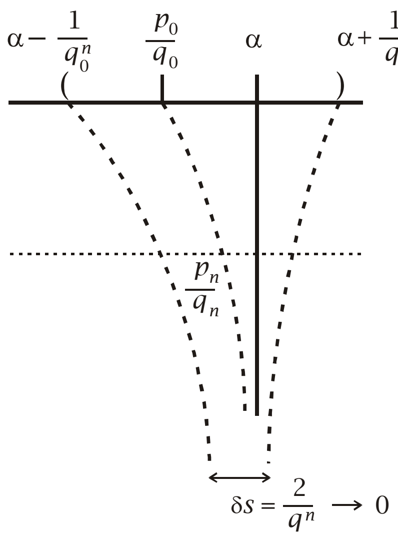


So for algebraic numbers, α, they are still bounded away from any near rational approximation. This relates to the result we noted earlier that every algebraic number may be correlated with some lattice point in **Fin**. There is some way to encode the construction of an algebraic irrational number using finite information. So algebraic numbers do not require encoding by actually infinite branches within the derived set.

16.2.2 Corollary to Liouville's theorem, transcendental Liouville numbers.

Let τ be a real number such that there exists an infinite sequence of rational numbers $\frac{p_n}{q_n}$ such that, $\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{(q_n)^n}$. Then τ must be a transcendental number. It is defined to be a Liouville number.

For Liouville numbers the situation is now reversed. Instead of being pushed apart from any rational approximation, there is a sequence of rational approximations to any Liouville number that is squeezed together with it.



16.2.3 Theorem, Every Liouville number is a Cohen real (+)

There will always be two nearest approximations to a Liouville number, α, one lying below it and one above, both with denominator q_n. These both converge on α without ever becoming identical to it. The theorem shows that they are squeezed together by an open set, here shown as $\left(\alpha - \frac{1}{(q_n)^n}, \alpha + \frac{1}{(q_n)^n}\right)$. In the limit the measure of this open set becomes incommensurable with zero. Although the two branches are still distinct, they cannot be separated by any measure. The Liouville number lies within this open set, and is not identical to either of its boundaries. Hence, any Liouville number is a Cohen real. The sequence $\frac{p_n}{q_n}$ in the corollary to Liouville's theorem is a sequence of boundary points, and comprises a meagre set, $\left\{\frac{p_n}{q_n} : n \in \mathbb{N}\right\}$. By Cohen forcing this sequence defines a transcendental number, α, not belonging to to this (or any other) meagre set; so a Liouville number is a Cohen real.

16.3 Transcendental numbers in general

A number is **transcendental** when it is not a zero of any nonzero polynomial in $\mathbb{Z}[t]$. We will examine polynomials $P(z) \in \mathbb{Z}[t]$ for which $|P(\xi)|$ is <u>nonzero</u> but as small as possible relative to the degree of $P(z)$ and the height of $P(z)$. If ξ is transcendental the zeros of polynomials that approximate ξ form a sequence that converge on ξ. A transcendental number corresponds to an infinite branch in the Cantor tree, 2^{ω}, which has no actually infinite sequence of all 1s or all 0s. (Such sequences are said to be **stagnant**.) A transcendental number is "infinitesimally" close to some rational approximation.

16.3.1 Definition, height of a polynomial

Let $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ be a polynomial; then the height is given by $h(P) = \max\{|a_0|, |a_1|, |a_2|, \ldots\}$.

16.3.2 Result

Let ξ be a transcendental number, $P(z)$ be a polynomial and α^* the zero of $P(z)$ closest to ξ. Then if $|P(\xi)| < \varepsilon$ then $|\varepsilon - a^*| < \varepsilon^{\frac{1}{N}}$.

<u>Proof</u>

Let $P(z) \in \mathbb{Z}[t], P(\xi) \neq 0$ be infinitely many polynomials. Suppose $|P(\xi)|$ is small then ξ must be very close to a zero of $P(z)$. Let $P(z)$ be factored as $P(z) = \prod_{n=1}^{N}(z - a_n)$ where $a_N \in \mathbb{Z}$ is the leading coefficient. Let $P(\xi) \neq 0$, then: -

$$0 < |a_N| \prod_{n=1}^{N} |\xi - a_n| = |P(\xi)|.$$

Let α^* denote the zero closest to ξ; so, $|\xi - \alpha^*| = \min\{|\xi - a_n|\} : n = 1, 2, \ldots, N$. Then

$$0 < |a_N| |\xi - \alpha^*|^N \leq |a_N| \prod_{n=1}^{N} |\xi - a_n| = |P(\xi)|$$

$$0 < |\xi - \alpha^*| \leq \frac{|P(\xi)|}{|a_N|} \leq |P(\xi)|^{\frac{1}{N}}$$

Thus, if $|P(\xi)| < \varepsilon$ then $|\varepsilon - a^*| < \varepsilon^{\frac{1}{N}}$.

16.3.3 Dirichlet's Theorem

Let $\tau \in \mathbb{R}$. Then there exists a constant $C = C(\tau)$ such that for any positive integer *H*, there exist integers *p* and *q*, where $0 < \max\{|p|, |q|\} \leq H$ such that: -

$$|\tau q - p| < \frac{C}{H}.$$

Further, if $H > C$, then $q \neq 0$.

<u>Proof</u>

The central step involves an application of the Dirchlet Pigeon-hole principle.

Begin by forming a collection of polynomials, $P_i(z) = a_1 z - a_0,\ 0 \leq a_1, a_0 \leq H,\ a_1, a_0 \in \mathbb{Z}$.

This can be seen as a "grid" of $(H+1)^2$ polynomials with $H+1$ points for values of the variable a_1 along one axis and $H+1$ for a_0. Then, for given $\tau \in \mathbb{R}$, we have $(H+1)^2$ functions, $P_i(\tau)$ from this grid onto $\mathbb{R}$. An application of the triangle inequality shows that these are all bounded: -

$|P_i(\tau)| \le B$ where $B = (1+|\tau|)H$.

So the values $P_i(\tau)$ all lie within a closed, bounded interval $S = [-B, B]$. Partition S into H^2 equal sub-intervals, then, by the Pigeon-hole Principle, for some $i \ne j$, we have both $P_i(\tau)$ and $P_j(\tau)$ contained in the same interval. Hence

$$|P_i(\tau) - P_j(\tau)| \le \frac{2B}{H} = \frac{2(1+|\tau|)}{H}$$

Then, let $P_i(\tau) = a_1 z + a_0,\ P_j(\tau) = a_1{}' z + a_0{}'$, and define

$p = a_0 - a_0{}',\ q = a_1 - a_1{}'$.

It follows that $\{|p|, |q|\} \le H$, and the other conditions in the theorem also can be shown. We also have $C = C(\tau) = 2(1-|\tau|)$.

16.3.4 Theorem, Diophantine condition for irrationality

For given $\tau \in \mathbb{R}$ and $H \in \mathbb{Z}$, let: -

$\Omega(\tau, H) = \min\{|P(\tau)| : P(z) = a_1 z + a_0 \in \mathbb{Z}[z]\},\ P(\tau) \ne 0,\ h(P) \le H$

where $h(P)$ is the height of the polynomial P. Then define $\omega(\tau, H)$ by: -

$\Omega(\tau, H) = H^{-\omega(\tau,H)}$

$\omega(\tau) = \lim\left\{\sup_{H\to\infty} \omega(\tau, H)\right\}$.

Then τ is irrational iff $\omega(\tau) \ne 0$.

<u>Proof</u>

This is a summary of the proof, for further detail see Burger and Tubbs [2004]. The result is a generalisation of the situation with Liouville numbers. Regarding Liouville's theorem, we have for a rational number $\alpha = \frac{r}{s}$ the result that $\left|\alpha - \frac{p}{s}\right| \le \frac{1}{Mq^n}$. This means that any rational approximation $\frac{p}{q}$ to a rational number $\alpha = \frac{r}{s}$ can be separated or "squeezed apart" from that number by an interval that is always of finite measure (provided the limit is not taken, which is given for a rational number.) For a Liouville number, which are transcendental numbers, the situation is reversed, and we have an interval, δs, of finite measure such that there always exists a rational approximation lying in this interval: $\left|\alpha - \frac{p}{q}\right| \leq \delta s$.

In generalising this we appear (initially) to weaken the criterion to cover merely irrational numbers. Thus, for a rational number $\alpha = \frac{r}{s}$ let $\frac{p}{q}$ be any rational approximation to it, not identical to it, then the interval $\left[\alpha - \frac{1}{sq}, \alpha + \frac{1}{sq}\right] \subset \left[\alpha - \frac{p}{q}, \alpha + \frac{p}{q}\right]$. Again we can always squeeze apart a rational number from any other near rational approximation to it. The notion that one interval is contained in the other is expressed in terms of Dirichlet's theorem: $\frac{1}{s} \leq |\alpha q - p|$ where $\alpha = \frac{r}{s}$. This follows from the Diophantine equation, $ax + by = c$ and when that has solutions. Here $\gcd(r,s) = 1$, so $rq + sp = 1$ has a solution.

Now for an irrational number τ, we can prove by means of the Dirichlet pigeonhole principle that the interval $\left[\tau - \frac{C}{H}, \tau + \frac{C}{H}\right]$ always contains another rational approximation to it. Specifically, H is a bound on the (algebraic) polynomials whose roots lie in an interval $\left[\tau - \frac{p}{q}, \tau + \frac{p}{q}\right]$ for integers $|p|,|q| \leq H$. Also, we specifically have $C(\tau) = 2(1 + |\tau|)$ so this is constructed. So given that we have at least one close approximation to τ lying inside the interval $\left[\tau - \frac{C}{H}, \tau + \frac{C}{H}\right]$ we can let $\Omega = \Omega(\tau, H)$ be the least such approximation. It is the least value $P(\tau)$ of all those polynomials with values close to τ. The theorem guarantees that this approximation lies within the interval $\left[\tau - \frac{C}{H}, \tau + \frac{C}{H}\right]$. To obtain the criterion for an irrational number explicitly, we define $\omega = \omega(\tau, H)$ by $\Omega = \frac{1}{H^{\omega}} = H^{-\omega}$. This is simply a recoding of the information contained in Ω, and by taking logs we have $\omega = -\frac{\log \Omega}{\log H}$. Then the criterion, $\Omega \leq \frac{C}{H}$, on taking logs, implies: -

$$-\omega \log H \leq \log\left(CH^{-1}\right)$$

$$-\omega \leq \frac{\log C}{\log H} - 1$$

$$1 \leq \frac{\log C}{\log H} + \omega$$

Let $\omega_{\tau} = \omega(\tau) = \lim \left\{\sup_{H \to \infty} \omega(\tau, H)\right\}$.

When $H \to \infty$ we have $\log H \to \infty$; hence $\frac{\log C}{\log H} \to 0$. Therefore, when $H \to \infty$ we have $\omega_\tau \le 1$. This entails that for τ irrational, we have $\omega_\tau \ne 0$.

Using the preceding theorem it can be shown that τ rational implies $\omega_\tau = 0$. So the following equivalence is established: -

τ is irrational iff $\omega(\tau) \ne 0$

16.3.5 Summary of the Diophantine condition for irrationality

1. Let τ be an irrational number. By the Dirichlet Pigeonhole Principle the interval $\left[\tau - \frac{C}{H}, \tau + \frac{C}{H}\right]$ contains another rational approximation to it, where: -
   - (*i*) H is a bound on the (algebraic) polynomials whose roots lie in an interval $\left[\tau - \frac{p}{q}, \tau + \frac{p}{q}\right]$ for integers $|p|, |q| \le H$.
   - (*ii*) $C(\tau) = 2(1 + |\tau|)$.
2. Let $\Omega = \Omega(\tau, H)$ be the least such approximation. It is the least value $P(\tau)$ of all those polynomials with values close to τ.
3. Define $\omega = \omega(\tau, H)$ by $\Omega = \frac{1}{H^\omega} = H^{-\omega}$. This is simply a recoding of the information contained in Ω, and by taking logs we have explicitly, $\omega = -\frac{\log \Omega}{\log H}$.
4. Then $\Omega \le \frac{C}{H}$ implies $1 \le \frac{\log C}{\log H} + \omega$.
5. Let $\omega_\tau = \omega(\tau) = \lim \left\{ \sup_{H \to \infty} \omega(\tau, H) \right\}$.

   When $H \to \infty$ we have $\log H \to \infty$; hence $\frac{\log C}{\log H} \to 0$.

   Therefore, when $H \to \infty$ we have $\omega_\tau \le 1$. This entails that for τ irrational, we have $\omega_\tau \ne 0$.
6. Using the preceding theorem it can be shown that τ rational implies $\omega_\tau = 0$. So the following equivalence is established: -
   τ is irrational iff $\omega(\tau) \ne 0$.

We need to extend this theorem to provide a criterion for when a number is not merely irrational, but transcendental. Transcendental numbers are non-algebraic. This generalization is provided by the generalized version of Liouville's theorem and an extension of Dirichlet's theorem.

16.3.6 Generalized version of Liouville's theorem

Let α be an algebraic number of degree *d*, and let *N* be a positive integer. Let $\partial P \le N$ be the degree of a polynomial *P*, and $h(P)$ its height. Then there exists a positive constant $c = c(\alpha, N)$ for all $P(z) \in \mathbb{Z}[z]$, $\partial P \le N$ and $P(\alpha) \ne 0$, such that: -

$$\frac{c}{\left[h(P)\right]^{d-1}} \le |P(\alpha)|$$

A detailed proof may be found in Burger and Tubbs [2004]. A proof outline follows below. This theorem replicates the situation of Liouville's theorem for all algebraic numbers; so it means that any number not satisfying the condition $\frac{c}{\left[h(P)\right]^{d-1}} \le |P(\alpha)|$ is transcendental, not merely Liouville. It also generalises the most recent result above from rational to algebraic numbers, or, conversely, from irrational numbers to transcendental numbers. It may be observed in general that if α is an irrational but algebraic number, the minimum polynomial for it demonstrates that it may be encoded by finite information. This is because its continued fraction is recurring. Algebraic numbers have finite codings.

In the proof we start with an algebraic number α. So it has a minimum polynomial $f(z)$ of degree $d = \partial f$. So any polynomial of higher degree which has α as a zero has *f* as a factor. This in turn means that any algebraic approximation to α, say β, must have a minimum polynomial P with degree $K = \partial P \le d$. Then it can be shown that $|\beta - \alpha| = |P(\alpha)| \ge \frac{c}{\left[h(P)\right]^{d-1}}$. The proof is really just a lot of algebra with summations and products.

16.3.7 Definition, "conjugates"

The zeros of the minimum polynomial for α are said to be its conjugates.

16.3.8 Proof outline of the generalized Liouville Theorem

Let $f(z) = \sum_{m=0}^{d} a_m z^m \in \mathbb{Z}[z]$ be the minimum polynomial for α.

Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be the conjugates of α, where $\alpha = \alpha_1$. Then we may write: -

$$f(z) = a_d \prod_{m=1}^{d} (z - \alpha_m).$$

Now suppose we have $P(z) = \sum_{k=0}^{K} b_k z^k$ where $K \le N$, and let $\beta_1, \beta_2, \dots, \beta_K$ be its zeros.

Then, $P(z) = b_K \prod_{k=1}^{K} (z - \beta_k)$.

Then, after some algebra we have: $P(\alpha) \ne 0$ implies $\alpha_m \ne \beta_k$ for all *m*, *k*.

Consider: -

$$\prod_{k=1}^{k}\prod_{m=1}^{d}|\alpha_m-\beta_k| = \prod_{k=1}^{k}\frac{1}{a_d}|f(\beta_k)| = \frac{1}{a_d}\prod_{k=1}^{k}|f(\beta_k)| \text{ using } f(z)=\sum_{m=0}^{d}a_m z^m \in \mathbb{Z}[z]$$

But similarly: -

$$\prod_{k=1}^{k}\prod_{m=1}^{d}|\alpha_m-\beta_k| = \prod_{m=1}^{d}\prod_{k=1}^{K}|\beta_k-\alpha_m| = \prod_{m=1}^{d}\left|\frac{1}{b_K}P(\alpha_m)\right| = \frac{1}{|b_K|^d}\prod_{m=1}^{d}|P(\alpha_m)| = \frac{1}{|b_K|^d}|P(\alpha)|\prod_{m=2}^{d}|P(\alpha_m)|$$

since $P(z)=\sum_{k=0}^{K}b_k z^k$ and $\alpha=\alpha_1$.

Putting the two together we get: -

$$P(\alpha)=\frac{|b_K|^d\prod_{k=1}^{k}|f(\beta_k)|}{(a_d)^K\prod_{m=2}^{d}|P(\alpha_m)|}.$$

It may be shown that the numerator here is a positive integer, so this gives: -

$$P(\alpha)\geq\frac{1}{(a_d)^K\prod_{m=2}^{d}|P(\alpha_m)|}$$

Furthermore, $|P(\alpha_m)|=\sum_{k=0}^{K}\left|b_K(\alpha_m)^K\right| \leq \sum_{k=0}^{K}|b_K|\left|\alpha_m{}^K\right| \leq h(P)\left(1+|\alpha_m|+|\alpha_m|^2+\dots+|\alpha_m|^K\right)$.

Putting $A=\max\{|\alpha_1|,|\alpha_2|,\dots,|\alpha_d|\}$ and $h(P)$ as the height of *P* as usual; we have: -

$$\frac{c}{[h(P)]^{d-1}}\leq|P(\alpha)|$$

where $c=\frac{1}{(a_d)^N}\cdot\frac{1}{\left(1+A+A^2+\dots+A^n\right)^{d-1}}$.

16.3.9 Theorem, Extension of Dirichlet's Theorem

Let ξ be a complex number and *N* a positive integer. Then there exists a constant $C=C(\xi,N)$ such that for any positive integer *H*, there exists a non-zero polynomial $P(z)\in\mathbb{Z}[z]$ with degree $\partial P\leq N$ and $h(P)\leq H$ such that: -

$$|P(\xi)|<\frac{C}{H^{\sqrt{N-1}}}.$$

16.3.10 Summary of the proof

A detailed proof may be found in Burger and Tubbs [2004].

From the proof we see that the constant in this formula is given by: -

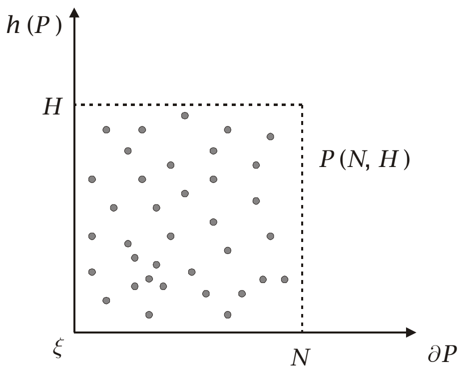
$$C=C(\xi,N)=6\sqrt{2}\left(1+|\xi|+|\xi|^2+\dots+|\xi|^N\right).$$

Let ξ be a complex number and let *H* and *N* be positive integers.

1. Define

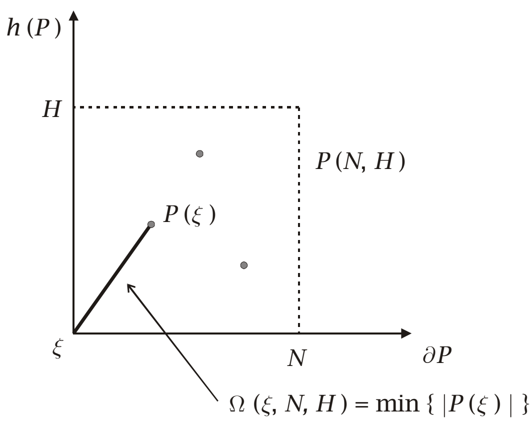
$$\underline{P}(N,H)=\{P(z)\in\mathbb{Z}[z]:\partial P\leq N \text{ and } h(P)\leq H\}$$

This is the set of polynomials with degree $\partial P \leq N$ and height $h(P) \leq H$. Let us think of ∂P and $h(P)$ as independent variables spanning $\mathbb{Z}^2$, so that *N*, *H* define a subspace, and $\underline{P}(N,H) = \{P(z) \in \mathbb{Z}[z] : \partial P \leq N \text{ and } h(P) \leq H\}$ is a subspace of the ring of all polynomials, $\mathbb{Z}[t]$. For given ξ, each of polynomial in $\underline{P}(N,H)$ takes a given value. We may think of these as points in the space: $\{P(\xi) : P(z) \in \underline{P}(N,H)\}$.



2. In the above diagram we think of the points as values of $P(\xi)$. If ξ is transcendental, then $P(\xi) \neq 0$. Since $P(\alpha) = 0$ implies that α is a root of *P*, then we know that if ξ has a very close polynomial approximation, *P*, then $|P(\xi)|$ is close to zero. The condition $|P(\xi)| < \dfrac{C}{H^{\sqrt{N-1}}}$ gives an upper bound to $|P(\xi)|$ and proves that every number has a rational approximation. So we then take the best approximation to ξ that is available within the region $\underline{P}(N,H)$. This is give by: -

$\Omega(\xi, N, H) = \min\{|P(\xi)| : P(z) \in \underline{P}(N,H), P(\xi) \neq 0\}$



3. As we let H and N grow, the density of the approximations to ξ increases and we get a sequence of improving approximations to ξ, in the sense that $|P(\xi)|$ gets closer and closer to zero. However, we recast the information given in $\Omega(\xi,N,H)$ by defining: -

$$\frac{1}{H^{\omega N}} = \Omega$$

Or in full, let $\omega(\xi,N,H)$ be given by, $\Omega(\xi,N,H) = \frac{1}{H^{\omega(\xi,N,H)N}}$.

Then by taking logs we get: -

$$\omega(\xi,N,H) = -\frac{\log\Omega}{N\log H}.$$

What this means is that as $\Omega(\xi,N,H)$ gets smaller, $\omega(\xi,N,H)$ gets bigger. That is why at the next stage we are looking at the supremum of $\omega(\xi,N,H)$ as opposed to the infimum.

4. We need to take limits in this expression. The theorem allows us to do this, we successively define: -

$$\omega(\xi,N) = \lim\left\{\sup_{H\to\infty}\omega(\xi,N,H)\right\}$$

$$\omega(\xi) = \lim\left\{\sup_{N\to\infty}\omega(\xi,N)\right\}.$$

5. Then, the number ξ is transcendental iff $\omega(\xi) \neq 0$.

The criterion $\omega(\xi) \neq 0$ affirms that the approximation $\Omega(\xi,N,H)$ has gotten so close to ξ that the interval $|P(\xi)|$ is incommensurable with zero. Adopting the principle or "axiom" that extensions are never annihilated [Section 10.4.6.] and also in accordance with the Axiom of Completeness, what this means is that $|P(\xi)| < \varepsilon$ for all $\varepsilon \in \mathbb{R}$. So $|P(\xi)|$ is still an extension, it is simply that we cannot measure its size. All transcendental numbers lie within such extensions. We have already seen that Liouville numbers, which are Cohen numbers ξ, are boundary points lying in open intervals of size $|P(\xi)| < \varepsilon$. Our main task now is to track down the amoeba reals and identify what kind of transcendental numbers they are.

16.3.11 Definition

Let $\nu(\xi)$ for the least positive integer N for which $\omega(\xi,N) = \infty$.

If $\omega(\xi,N)$ is finite for all N then $\nu(\xi) = \infty$ by convention.

If $\omega(\xi)$ is finite, then $\nu(\xi) = \infty$, but if $\omega(\xi) = \infty$ then $\nu(\xi)$ may be either finite or infinite.

16.3.12 The Mahler classification

A number	$\omega(\xi)=0$	$\nu(\xi)=\infty$
S-number	$0<\omega(\xi)<\infty$	$\nu(\xi)=\infty$
T-number	$\omega(\xi)=\infty$	$\nu(\xi)<\infty$
U –number	$\omega(\xi)=\infty$	$\nu(\xi)=\infty$

We claim that the S numbers are amoeba reals. The following theorem is vital to demonstrating the validity of this claim.

16.3.13 Theorem for *S* numbers

Let $\xi \in \mathbb{C}$ be a transcendental number. Then $0<\omega(\xi)<\infty$ iff there exists a real number $\rho>0$ such that for each integer $N\geq 1$, there exists a constant $c'=c'(\xi,N)>0$ such that for all integers $H\geq 1$ and all polynomials $P(z)\in \underline{P}(H,N)$, the inequality, $\frac{c'}{H^{\rho N}}<\left|P(\xi)\right|$ holds.

A proof may be found in Burger and Tubbs [2004].

16.3.14 The construction of the constant in the above theorem

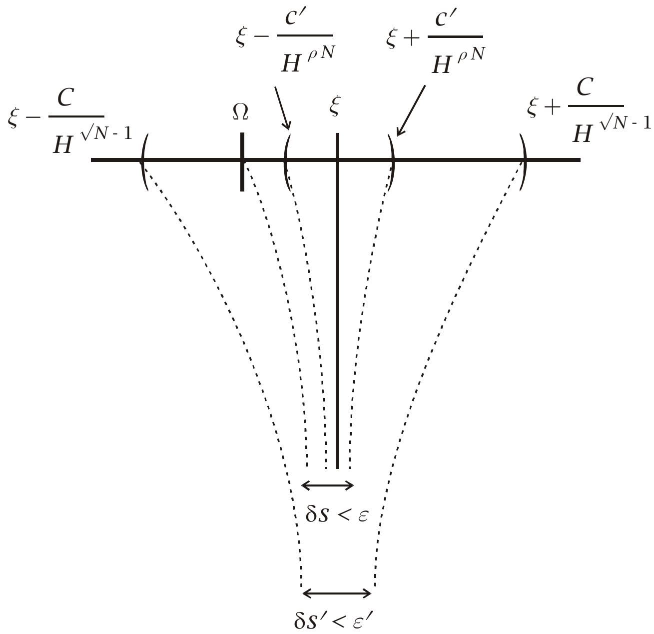
The constant is explicitly constructed as follows: -

1. Since $0<\omega(\xi)<\infty$ there exists a $\rho>0,\ \rho\in\mathbb{R}$ such that $\omega(\xi,N)<\rho$.
2. Then for fixed N and all but finitely many H's we have, $\frac{1}{H^{\rho N}}\leq \Omega(\xi,N,H)\leq\left|P(\xi)\right|$.
3. Let there be a list of the exceptions for H to the rule $\frac{1}{H^{\rho N}}\leq \Omega(\xi,N,H)\leq\left|P(\xi)\right|$. Write this list as $H_1<H_2<\ldots<H_L$ for some finite L. Let $m=\min\left\{\Omega(\xi,N,H):l=1,2,\ldots,L\right\}$.
4. Then $c'=c'(\xi,N)=\min\left\{1,\frac{m}{2}H_L^{\rho N}\right\}$.

This theorem asserts that the growth rate in the Extension to Dirichlet's Theorem cannot be improved. A constant multiple of N is the best possible exponent on H iff $0<\omega(\xi)<\infty$. S-numbers are precisely those transcendental numbers ξ for which $\left|P(\xi)\right|$ cannot be made substantially smaller than the upper bound in the Dirichlet Extension Theorem, which was found by application of the Pigeonhole principle. Recall, that that upper bound was given by,

$\left|P(\xi)\right| < \frac{C}{H^{\sqrt{N-1}}}$. Combining this with the inner bound of the theorem for *S* numbers, we obtain the inequality: -

$\frac{C}{H^{\sqrt{N-1}}} > \left|P(\xi)\right| > \frac{c'}{H^{\rho N}}$.



The diagram illustrates that in the limit, as $N \to \infty$, the transcendental *S* number, ξ, is identified with the interval, $\left(\xi - \frac{c'}{H^{\rho N}}, \xi + \frac{c'}{H^{\rho N}}\right)$, and is itself squeezed into a measure incommensurable with zero, $\left(\xi - \frac{C}{H^{\sqrt{N-1}}}, \xi + \frac{C}{H^{\sqrt{N-1}}}\right)$. On the principle that extensions are never destroyed any *S* number is an extension point.

16.3.14 Theorem, *S* numbers are amoeba reals (+)

We have just proven this result.

The number *e* has been proven to be an *S* number. The classification of π is unknown. Burger and Tubbs remark that the existence of *T*-numbers was only established in 1968 by Wolfgang Schmidt.

16.3.15 Further classification

It remains to classify the other numbers. Here we cite a series of results for this purpose.

1. Algebraic numbers are *A* numbers.

2. Liouville numbers are U numbers.
3. "T-numbers are precisely those ξ's for which we have infinitely many polynomials $P(z) \in \mathbb{Z}[z]$ such that $|P(\xi)|$ can be made *substantially* smaller than the upper bound of Theorem 6.5 but the degrees of those amazing polynomials must be unbounded." (The reference is to Burger and Tubbs [2004] Theorem 6.9 page 162.]

We have already established that the Liouville numbers are Cohen reals. The result (3) above also makes T numbers into Cohen reals, a conclusion that will shortly be confirmed when we consider the measure of the sets of these separate categories of number.

16.4 Mahler's number

16.4.1 Theorem, relation between the point at infinity and Mahler's number (+)

There exists a bijection between initial segments of λ and Mahler's number, $M = 0.12345678910111213141516...$.

Meaning of segment

Suppose $x = 0.0000123...$. Then 0.00001 is a **segment** of *x*. (In this case it is also an initial segment.)

Proof outline

λ well-orders the potentially infinite antichain $\mu = \{\{1\},\{2\},\{3\}, \ ...\}$. So λ must have at least as much structure as $\mathbb{N}$, since every element of μ is in one-one correspondence, in order, with an element $n \in \mathbb{N}$.

Illustration

λ is a transcendental number representing the next nearest number to 1 in the unit interval. Likewise, there is a number like λ that is the next nearest number to 0 in the unit interval. We may think of this number as a member of ω^ω that comprises an infinite sequence of 0s followed by Maher's number: -

0.0000 ... 00012345678910111213141516 ...

It is any number with an equivalent structure: for example, an infinite sequence of 0's followed by a 1, followed by an infinite sequence of 0's followed by a 2, and so forth. Every natural number must follow in order in the sequence somewhere. Its order type is $< \omega_1$. We think of λ as the result of subtracting this number from 1.

Proof of the theorem

Mahler's number is $M = 0.12345678910111213141516...$

M is the infinite sequence in ω^ω given by: $n \to n$.

$\mu = \{\{1\},\{2\},\{3\}, \ ...\}$ is the potentially infinite antichain of unordered elements.

By the Nested Interval Principle [Section 4.1 above] and by the fact that μ is not a skeleton for $[0,1]$, $\{\lambda\}$ is not empty. Therefore, $\lambda \neq \varnothing$. λ cannot be an algebraic number, for then it would be a meet in **Fin**. So λ is a transcendental number, or a set of transcendental numbers. If it is a set of transcendental numbers, let λ be the least transcendental number in $\{\lambda\}$.

There exists a well-ordering of μ external to μ. Therefore, there is a well-ordered set that well-orders μ. This well-ordering is an intrinsic property of the continuum. Therefore, it is intrinsic to the skeleton. Since $\{\lambda\}$ is the complement in $\mathbb{N}_\lambda$ of μ: $\{\lambda\} = \mu'$, then $\{\lambda\}$ is a well-ordered set that well-orders μ. Since μ is a collection of atoms each of which may be placed in one-one correspondence with $\mathbb{N}$ and $\{\lambda\}$ well-orders μ, then there is a map of segments of λ on to $\mathbb{N}$. Denote the nonzero segments of λ by $s_1(\lambda), s_2(\lambda), \ldots, s_n(\lambda), \ldots$. Then $s_1(\lambda) = 1, s_2(\lambda) = 2, \ldots, s_n(\lambda) = n, \ldots$.

Then there is bijection, $n = s_n(\lambda) \to n$. That displays it as a bijection onto Mahler's number.

16.4.2 Corollary

$\lambda \in \omega^\omega$ is a Cohen real.

<u>Proof</u>

$\{\lambda\}$ represents the neighbourhood of 1 in the unit interval. It is open interval in $[0,1]$. Hence it is dense in itself. Its measure is $m(\{\lambda\}) = 0$. Therefore $\{\lambda\}$ is a null set. Since λ belongs to a null set it cannot be an Amoeba real; since all transcendental numbers are either Cohen or amoeba, it is a Cohen real.

17. Posets and forcings in general

17.1 The abstract character of forcing

Up to this moment we have used two different though related characterisations of the forcing arguments. Both concern conditions given in a poset, $\mathbb{P}$, described in a language *K*, which corresponds to or "talks about" the Cantor set, or more specifically, the derived set of the one-point compactification of the continuum. We first described forcing in very abstract terms; for example, we defined Cohen forcing to be the process of extending a potentially infinite branch within the lattice of boundary points to an actually infinite. We later gave a more concrete description of forcing in terms of posets in a forcing language corresponding to lattice extensions: -

1. Within the forcing language *K* we describe a poset $\mathbb{P}$ to possess an increasing chain of conditions, $p_0 \le p_1 \le \ldots \le p_k \le \ldots$, given by set inclusion, $p_k \le p_{k+1}$ iff $p_k \subseteq p_{k+1}$. These correspond to a chain of filters in the lattice, $\text{filter}(p_0) \subseteq \text{filter}(p_1) \subseteq \ldots$

$\ldots \subseteq \text{filter}(p_k) \subseteq \ldots$, and since at the finite stage these are all principal, to a descending chain of lattice points, $\{p_0\} \geq \{p_1\} \geq \ldots \geq \{p_k\} \geq \ldots$. These are conceived as belonging to ever-increasing embeddings of finite lattices, all contained with the lattice of all finite subsets of ω, denoted **Fin**.

2. At every stage there is a forcing rule that specifies that a given member of the poset, $q \in \mathbb{P}$, cannot be given by any finite member of the poset: $q \neq p_n \in \mathbb{P}$ for any $n \in \mathbb{N}$. The assumption that it does leads to a formal contradiction. This may be further expressed by saying at at the level $n \in \mathbb{N}$ there exists in the poset, $\mathbb{P}$, incompatible elements, $p_n, p_n' \in \mathbb{P}$ such that $p_n \cap p_n' = \varnothing$.
3. We then postulate that corresponding to $q \in \mathbb{P}$ there is a filter, $\text{filter}(q)$, which is a family of subsets within the Cantor set, 2^ω.
4. Since $\text{filter}(q)$ cannot be constructed at any finite level $q \neq p_n \in \mathbb{P}$, it follows that we must extend the chain in $\mathbb{P}$ to a chain of ordinal length actually infinite. In all cases so far considered, this is of order type ω.

To convert this description into a more abstract one, we consider a ground model *M*, which is a countable transitive model of a finite part of ZFC. *G* will be a generic filter, and in general, *G* is not a family of subsets contained in *M*. Hence $M[G]$ will be a generic extension of *M*. The filter *G* will be defined by a poset, $\mathbb{P}$, which, since the discussion almost universally takes place within the context of first-order set theory, is actually a family of sets contained in the ground model *M*. This enables *M* to describe and define its own generic extensions. In order to define a generic filter, *G*, *t*he essential properties of the poset, $\mathbb{P}$, are: -

(a) That the conditions in $\mathbb{P}$ form an infinite chain; in this case $\mathbb{P}$ is said to be **dense**.

(b) That the condition *q* lies in $\mathbb{P}$ and cannot be reached by any finite chain of conditions starting from some initial condition $p_0 \in \mathbb{P}$.

(c) That at every level in $\mathbb{P}$ there is an antichain, or a pair of incompatible conditions. This means that in order to construct the filter, $G = \text{filter}(q)$, one must complete an actual infinity.

17.1.1 Definition, dense partial order

Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ iff $(\forall p \in \mathbb{P})(\exists q \leq p)(q \in D)$. (Kunen [1980] p. 53)

17.1.2 Definition, generic

Let $\mathbb{P}$ be a partial order. *G* is said to be $\mathbb{P}$-generic over *M* iff *G* is a filter on $\mathbb{P}$ and for all dense $D \subset \mathbb{P}, D \in M \rightarrow G \cap D \neq 0$. (Kunen [1980] Chapter 7, Def. 2.2, p. 186)

17.2 Relative consistency results

Kunen explains the idea behind relative consistency results: "Let *M* be a countable transitive model for ZFC. If $\langle \mathbb{P}, \leq \rangle$ is a partial order … and $\langle \mathbb{P}, \leq \rangle \in M$, then $\langle \mathbb{P}, \leq \rangle$ will yield a method of obtaining a generic extension, *N*, of *M*, which is also a model of ZFC. By varying $\langle \mathbb{P}, \leq \rangle$, we shall be able to produce a variety of relative consistency results." (Kunen [1980] Chapter 7, Sec. 2, p. 186)

17.2.1 Definition, Martin's Axiom

$\mathrm{MA}(\kappa)$ is the statement: Whenever $(P, \leq)$ is a non-empty c.c.c. partial order, and **D** is a family of $\leq \kappa$ dense subsets of *P*, then there is a filter *G* in *P* such that $(\forall D \in \mathbf{D})(G \cap D \neq 0)$. MA is the statement $(\forall \kappa < 2^{\omega})(\mathrm{MA}(\kappa))$.

17.2.2 Results

1. $\mathrm{MA}(\omega)$ is true.
2. $\mathrm{MA}(2^{\omega})$ is false.

<u>Outline of proof of (1)</u>

$\mathrm{MA}(\omega)$ follows from the Completeness Axiom. The result, $\mathrm{MA}(\omega)$, is regarded as a general result in ZFC about countable models: If *M* is countable and $p \in \mathbb{P}$, then there is a *G* which is $\mathbb{P}$-generic over *M* such that $p \in G$. (Kunen [1980] Lemma 2.6 p. 55)[34]

<u>Out line of proof of (2)</u>

It is an application of Cantor's anti-diagonalisation argument. We begin by assuming that there is a denumerable list of all functions $h : \omega \to 2$. This constitutes a partial order: $h_1, h_2, \ldots h_n, \ldots$. So we can define a dense subset of this partial order, E_h. Then by $\mathrm{MA}(2^{\omega})$ this defines a generic set *G*. This in turn defines a function $f_G : \omega \to 2$ contradicting the assumption that we had listed them all.

$\mathrm{MA}(\kappa)$ only becomes relevant if there is a cardinal, $\aleph_0 < \kappa < 2^{\aleph_0}$. We will subsequently demonstrate that the Completeness Axiom entails $2^{\aleph_0} = \aleph_1$ [Section 18], so this also entails $\mathrm{MA}(\aleph_1)$ is false. To be explicit, this is the <u>conditional statement</u>: if the Completeness Axiom is valid, then $\mathrm{MA}(\aleph_1)$ is false. What set theory has done is to implicitly open up the

[34] However, I have some problems with the proof in Kunen, which seems to be circular by assuming the existence of the filter *G* which is is intended to prove. $\mathrm{MA}(\omega)$ is equivalent to allowing the completion of a potentially infinite sequence to an actually infinite one, thus defining a limit. This is precisely what the Completeness Axiom allows us to do. The truth of $\mathrm{MA}(\omega)$ in ZFC set theory is an example of the implicit use of the Completeness Axiom and also arises from identifying $\mathbb{N} = \omega$.

possibility of defining other structures on the continuum. Ultimately, the validity of the Completeness Axiom will become an empirical question. Within first-order set theory all sorts of alternative models are possible. Kunen [1980] explains this point as follows: "The particular axioms of set theory that $M[G]$ satisfies beyond ZFC will be very sensitive to the combinatorial properties satisfied by $\mathbb{P}$ in *M*; most of these properties are *not* absolute. For example, consider the c.c.c. If *M* is a c.t.m. and $\mathbb{P} \in M$, then in **V**, $\mathbb{P}$ is countable and thus trivially has the c.c.c. But $\mathbb{P}$ may well fail to have the c.c.c. in *M*."

Here we will give two illustrations from Kunen's work of the use of iterative Cohen forcing to produce alternative models of the continuum. The idea is to give the "flavour" of the concept rather than the details, since our topic is the question of what follows specifically from the Axiom of Completeness.

17.2.3 Examples of forcings

1. $\mathbb{P} = \mathrm{Fn}(\omega,\kappa)$

   Here $\mathrm{Fn}(\omega,\kappa)$ denotes the set of all partial functions from ω into κ. Since κ is uncountable, this poset does not satisfy the countable chain condition.
   In the ground model, *M*, we have $\kappa > \omega$.

   In the generic extension $M[G]$ we have $\kappa = \omega$.

   This forcing is used to show that the notion of cardinality is not absolute for *M*, or absolute in ZFC. A set may have cardinality $\kappa > \omega$ in the ground model, and cardinality $\kappa = \omega$ in the extension, $M[G]$.

2. $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$

   This satisfies the countable chain condition.

   In the ground model, *M*, we have $\kappa > \omega$ and κ, 2^{ω} are incomparable.

   In the generic extension $M[G]$ we have $\omega < \kappa = 2^{\omega}$.

   Since κ can be anything, this falsifies CH.

The obvious point to make about these forcings is that they do not directly have any bearing on the second-order Completeness Axiom, which tells us how to complete the arithmetical continuum: starting with a dense subset, $\mathbb{Q}$, we create a collection of all linear orderings of elements of $\mathbb{Q}$, and then identify the members this set with the reals. The partial order that is being considered is just $\mathbb{P} = \mathrm{Fn}(\omega,2)$, which is Cohen forcing, and the completion in question concerns the closure of potentially infinite binary sequences by actually infinite sequences. It is the step from the potential to the actual, from $\mathbb{N}$ to ω that is considered, which implies $\mathrm{MA}(\omega)$. But further consideration reveals that Cohen forcing constructs only the boundary points of the continuum, and therefore leaves over another collection of extension points, since boundaries never diminish n measure of an interval. So we have in addition to Cohen reals, amoeba reals. Amoeba reals are not generated by any forcing in the same sense in which other reals are – there is no poset to which an individual amoeba real corresponds. The Completeness Axiom implicitly rests on a second primitive notion of extension, namely, that space is always comprised of spaces. Therefore, we additionally need amoeba reals generated by actually infinite nested sequences of open intervals that are now identified with *S* numbers. We have the following theorem.

17.2.4 Forcing theorem (+)

All transcendental numbers correspond to ultrafilters (functions) of **Fin** and **Cofin** embedded within the Derived set, $2^{\mathbb{N}_\lambda}$, of the actually infinite one-point compactification of the arithmetical continuum.

Remark

We have already shown that all Liouville numbers and the number *e* are generic sets.

Proof

Let ξ be a transcendental number. In the Extended Dirichlet Theorem, we construct a series of approximations $P(\xi)_{N,H}$ where

$\Omega(\xi,N,H)=\min\left\{\left|P(\xi)\right|:P(z)\in\underline{P}(N,H),P(\xi)\neq 0\right\}$

and $\omega(\xi,N,H)=-\dfrac{\log\Omega}{N\log H}$. Since ξ is transcendental, we also have $\omega(\xi)\neq 0$. $P(\xi)_{N,H}$, is a series of algebraic codes that correspond to lattice points within the set of all finite subsets of ω: $\mathrm{Fin}=\mathrm{F}(\omega)\cong 2^{<\omega}$ or the set of all co-finite subsets of ω: $\mathrm{Cofin}=\mathrm{C}(\omega)\cong 2^{<\omega}$. The conditions,

$\Omega(\xi,N,H)=\min\left\{\left|P(\xi)\right|:P(z)\in\underline{P}(N,H),P(\xi)\neq 0\right\}$

$\omega(\xi,N)=\lim\left\{\sup_{H\to\infty}\omega(\xi,N,H)\right\}$

$\omega(\xi)=\lim\left\{\sup_{N\to\infty}\omega(\xi,N)\right\}$

together constitute a set of forcing conditions and define a poset, $\mathbb{P}(\xi)$. This poset is open, dense and at each finite level the assumption that the number ξ is defined by a polynomial immediately leads to the contradiction: $\omega(\xi)=0$. The conditions, $\omega(\xi)=0$ and $\omega(\xi)\neq 0$ determine incompatible lattice points at any finite level. Therefore, the poset $\mathbb{P}(\xi)$ defines a generic ultrafilter within **Fin** or **Cofin**.

17.3 The dependency of the Mahler classification

Liouville's theorem depends explicitly on the Mean Value theorem which in turn rests on the Completeness Axiom. The two critical principles used throughout these arguments, leading to the Generalized Dirichlet Theorem are: -

1. The second-order Axiom of Completeness

2. The Dirichlet Pigeonhole Principle.

The Mahler classification is a consequence of these two principles. We may summarise the entire logic of the arguments of this paper up to this point as follows: -

$$\left.\begin{array}{c}\text{Dirichlet Pigeon-hole Principle}\\ \text{Axiom of Completeness}\end{array}\right\}\Rightarrow\begin{cases}\text{1-point compactification of the skeleton of the real line}\\ \text{Mahler classification of transcendental numbers}\\ \text{Forcing theorem}\end{cases}$$

17.4 The Mahler classification and measure

The following definition comes from Burger and Tubbs [2004] (Section 6.6, p.167)

17.4.1 Definition, "almost all"

We say that *almost all* complex numbers are in a particular set X if the complement $\mathbb{C}-X$ is a set of measure zero.

17.4.2 Theorem

Almost all numbers are transcendental.

Proof

Burgess and Tubs remark that, "the statement of the theorem is equivalent to establishing that the set of all algebraic numbers is a set of measure zero." We have already proven above that the set of all algebraic numbers is a set of cardinality $\aleph_0$.

The following are results proven in Burgess and Tubbs [2004].

17.4.3 Theorem

1. The set of U numbers is a set of measure zero.
2. The set of T numbers is a set of measure zero.

17.4.4 Corollary

Almost all numbers are S-numbers.

The proof of the important corollary is non-constructive. While the other proofs show that the sets of A, U and T numbers have measure zero, the proof that the set of S numbers have measure 1 (in the unit interval) works backwards from the fact that the measure of the unit interval is 1. That is: -

Proof of the corollary

$$\mu^*(A)+\mu^*(S)+\mu^*(T)+\mu^*(U)=\mu^*([0,1])=1$$
$$\mu^*(A)+\mu^*(T)+\mu^*(U)=0$$
$$\mu^*(S)=1$$

(Here μ^* represents measure.)

For this reason the definition "almost all" is "almost" a mild abuse of language. Or rather, it is a question of understanding thoroughly what "almost all" means. It does not mean that there are more S numbers than any of the other numbers. The cardinality of the other sets combined is continuum, which can be seen from the manner in which they care constructed from the Cantor tree. Likewise, the cardinality of the S numbers is also continuum. (The set of S –numbers is proven indirectly to be of second category; but the set of Liouville numbers is second category; so we cannot say there are more S numbers than Liouville numbers; from the cardinality point-of-view, there are just as many of the one kind as there are of the other.) So "almost all" does not indicate "larger" in the sense of cardinality. This argument simply confirms that the S numbers (the amoeba reals) are not boundary points but extensions. [See above, Theorem 16.3.14.] It confirms that on the arithmetical continuum subject to the Axiom of Completeness the principle of the indestructibility of extensions is a consequence.

18. The Derived set and the continnum hypothesis

18.1 Derivation of the Continuum Hypothesis from the Axiom of Completeness

Here we prove that the Completeness Axiom entails the Continuum Hypothesis. However, this does not prove the Continuum Hypothesis in an absolute sense. It does show that it is implied by the natural conception of completing the rational numbers provided by the Completeness Axiom, which is used time and again in all analysis, implicitly or explicitly; we mean, that analysis that takes place outside the formulation of first-order set theory.

18.1.1 Lemma [Halbeisen [2012]

For every ordinal $\alpha \in \omega_1$ there is a set of rationals $Q_\alpha \subseteq \mathbb{Q} \cap (0,1)$ and a bijection $h_\alpha : \alpha \to Q_\alpha$ such that for all $\beta, \beta' \in \alpha,\ \beta \in \beta' \Leftrightarrow h_\alpha(\beta) < h_\alpha(\beta')$.

Proof in Halbeisen [2012], Lemma 4.10 p. 77.

Halbeisen remarks that this means that for any $\alpha \in \omega_1$ there is a set of rationals, $Q_\alpha \subseteq \mathbb{Q} \cap (0,1)$ such that $\eta(Q_\alpha, <) = \alpha$ where η is the order type of the well-ordering of Q_α. Since the lemma states that the function is a bijection, this makes the set Q_α unique: that is, to each well-ordering of the rationals there corresponds a unique ordinal $\alpha \in \omega_1$ and conversely. The proof gives a method of constructing this bijection. Hence we have: -

18.1.2 Corollary

ω_1 is the set of order types of well-orderings of $\mathbb{Q}$.

From this Halbeisen [2012] shows that there is a surjection from the open interval $(0,1)$ onto ω_1; hence, $\aleph_1 \leq 2^{\aleph_0}$.

18.1.3 Theorem, the Completeness Axiom entails the Continuum Hypothesis (+)

Any well ordering is an element of $2^{\mathbb{Q}} = \mathbf{P}(\mathbb{Q})$, and conversely. However, if $\mathbb{Q}$ is regarded as a potential infinity, its power set is also only potentially infinite, and hence of cardinality $\aleph_0$. So in order to obtain an actually infinite skeleton of the continuum, we must treat $\mathbb{Q}$ as a completed infinity – not merely potential one. We adjoin to $\mathbb{Q}$ an "infinite" element to obtain a model $\mathbb{Q}_\lambda = \mathbb{Q} \cup \{\mathbb{Q}\}$ by analogy with $\mathbb{N}_\lambda$. The lemma then gives us: -

$\aleph_1 = \text{card}(\omega_1) = \text{card}(\mathbf{2}^{\mathbb{Q}_\lambda})$.

But, with the one-point compactification of the continuum, we obtain a skeleton that completely determines the structure of the Cantor set. We obtain: -

$\mathfrak{c} = \text{card}(\mathbb{R}) = \text{card}(2^{\mathbb{N}_\infty})$

Now since we have $\mathbb{N}_\lambda \subset \mathbb{Q}_\lambda$, or alternatively, $\text{card}(\mathbb{N}_\lambda) = \text{card}(\mathbb{Q}_\lambda) = \aleph_0$, then $2^{\mathbb{N}_\lambda} \subseteq 2^{\mathbb{Q}_\lambda}$.

Whence $\mathfrak{c} = 2^{\aleph_0} = \text{card}(2^{\mathbb{N}_\infty}) \le \text{card}(2^{\mathbb{Q}_\infty}) = \omega_1$.

Hence, $2^\omega \le \omega_1$ which proves CH on the arithmetical continuum with skeleton $\mathbb{N}_\infty$ and also $2^{\mathbb{N}_\lambda} \cong 2^{\mathbb{Q}_\lambda}$.

The Dedekind Completeness Axiom states: Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound (supremum) in the set [Section 4.1]. If we replaced $\mathbb{R}$ here by $\mathbb{Q}$ then the continuum becomes the set of all well-orderings of $\mathbb{Q}$ and then by Corollary 18.1.2, ω_1 is the set of order types of well-orderings of $\mathbb{Q}$, so $\mathfrak{c} = \omega_1$. This is essentially the situation with the Cohen reals shown here to be the set of all well orderings of $\mathbb{Q}$. Although there are other amoeba reals in the continuum, the Axiom of Completeness entails the Heine-Borel theorem, and that in turn entails that we construct a one-point compactification for the skeleton of the continuum, and obtain the continuum as its derived set. The continuum cannot be null, so we must decompose the skeleton into two disjoints sets: one of boundary points and the other of intervals. Nonetheless, taken together, the two parts to the skeleton comprise a single skeleton, denoted $\mathbb{Q}_\lambda$. This has order type $\omega + 1$, and the continuum, generated as the Derived set, has cardinality $\aleph_1 = \text{card}(\omega_1) = \text{card}(\mathbf{2}^{\mathbb{Q}_\lambda}) = \text{card}(2^{\omega+1})$. Whence, $2^{\aleph_0} = \aleph_1$ or $\mathfrak{c} = \omega_1$. Another way of looking at this is that the continuum comprises three disjoints sets of reals: the algebraic reals of cardinality $\aleph_0$, the Cohen reals of cardinality $\mathfrak{c}$ and the amoeba reals, also of cardinality $\mathfrak{c}$. The Axiom of Completeness does not add more amoeba reals than Cohen reals, and for the Cohen reals we have $\text{card}(\text{Cohen}) = \omega_1$.

Fundamentally, the Completeness Axiom imposes the "simplest" structure admissible on the continuum subject to the following constraints: (a) real numbers are generated as limits of potentially infinite sequences from its dense subset, $\mathbb{Q}$; (b) the "Axiom" of the indestructibility of extensions.

18.1.4 On the need for AC

This proof of CH ultimately does not depend on the Axiom of Choice. One can use it as a hypothesis that can then be dropped. Reason. Once one has established $2^{\aleph_0} = \aleph_1$ as the correct cardinality of $\mathbb{N}_\infty$ then if we drop the well ordering of the infinite element $\lambda = \{\mathbb{N}\}$ the resultant structure has exactly the same number of lattice points after as before. We refer to the model $\mu_\infty = \mu \cup \{\lambda\}$ where μ is a potentially infinite un-ordered antichain that partitions the unit interval.

As the mapping $\eta : \omega_1 \to \mathbf{P}(\mathbb{Q})$ of the Lemma 18.1.1 is a bijection, it has an inverse: $\eta^{-1} : \mathbf{P}(\mathbb{Q}) \to \omega_1$. This gives a bijection of the Cohen reals onto ω_1: $\eta^{-1} : 2^\omega \to \omega_1$. Every element in $\mathbf{P}(\mathbb{Q})$ is a branch in $\mathbf{2}^\omega$. Given $\mathbf{P}(\mathbb{Q}) \cong \mathbf{2}^{\mathbb{N}_\lambda}$ which is entailed by the Axiom of Completeness, then $\mathbb{Q}_\lambda \sim \mathbb{N}_\lambda$ represents all atoms, both boundaries and extensions, and $\mathbf{P}(\mathbb{Q}) \cong \mathbf{2}^{\mathbb{N}_\lambda}$ is its Derived set. So this is the elusive bijection of all real numbers onto ω_1. The Completeness Axiom entails the Continuum Hypothesis.

18.2 The Derived set and the arithmetical continuum

The problem we address here is the issue of whether the Derived set is homeomorphic to the arithmetical continuum, $\mathbb{R}$. I have claimed that the Derived set, $\mathbf{2}^{\mathbb{N}_\lambda}$ is categorical for $\mathbb{R}$. In ZFC the Derived set is equinumerous to the Cantor set, $\mathbf{2}^\omega$, though the question of what additional axioms in ZFC are needed to bring about an isomorphism between them is still open. $\mathrm{MA}(\omega)$ is already a theorem of ZFC, but additionally we require $\neg\mathrm{MA}(\kappa)$, $\omega < \kappa \le \mathfrak{c} = 2^{\aleph_0}$.[35] It is a claim made here that the Cantor set is consistent with any number of interpretations of the continuum, but still we would expect the Derived set to inherit all the properties of the Cantor set that can be derived in ZFC – in particular, that it is nowhere dense and totally disconnected. But if that is so, how can the Derived set be a model of the continuum?

The resolution of this problem is found in the observation that the Cantor set is first category (meagre) in $\mathbb{R}$, but second category in itself. As a model of the continuum, the Derived set is second category in itself, and hence dense, connected, separable and continuous.

[35] I doubt whether any first-order theory could suffice to delineate the same model. It seems that the continuum can only be delineated in a second-order theory.

This possibility arises from the nature of the Second order reasoning that is employed here. Recall that the Axiom of Completeness is irreducibly second order. The nature of this background logic is as yet to be examined; to be sure I must presume that the arguments displayed here are consistent deductions in second order logic, possibly supplemented by natural language argument. Part of this implicit background machinery is the conclusion that second-order logic permits us to impose simultaneous structures on an object and to view it both internally and externally. Internally, the skeleton of the continuum is an unordered antichain; externally, it is a well-ordered chain in one-one correspondence with ω.

Another dual structure that is essential to this work emerges when we interpret the relationship between the set ω and the unit interval, $\mathbb{I} = [0,1]$. As a set ω is the actually infinite collection of all finite ordinals as well as its upper bound; it is discrete and totally disconnected. As an interpretation of the skeleton of the continuum it is a partition of the unit interval into ω segments each of which is a half-closed, half-open interval: $[q_1, q_2)$ where $q_1, q_2 \in \mathbb{Q}$. As such the skeleton is a connected space homeomorphic to the unit interval. We further decompose this skeleton into two sub-skeletons [Section 10.3]: -

$\mathbb{N}_A \cup \{\{\lambda\}\}$ The skeleton of atoms of boundary points corresponding to rational numbers, collectively both a null and meagre set, with cannoical representation: -
$\mathbb{N}_\lambda = \{\{1\},\{2\},\{3\}, \ldots\}$.
The atom $\{\lambda\}$, which represents the neighbourhood of 1, is not a member of this collection.

$\mathbb{N}_B \cup \mu$ The skeleton of co-atoms of the derived Cantor set of the continuum; these are open intervals and collectively neither null nor meagre sets. The co-skeletton has cannoncial representation: -
$\omega - \mathbb{N}_\lambda = \{\omega - \{1\}, \omega - \{2\}, \omega - \{3\}, \ldots\}$.
The co-atom $\mu = \{\lambda\}' = \omega - \{\lambda\}$ is not a member of this collection and represents the neighbourhood of 0 in the interval $[0,1]$.

Both sub-skeletons are isomorphic to ω, and indistinguishable from it in ZFC; in second-order logic they are very different. The power set of $\mathbb{N}_A \cup \{\{\lambda\}\}$ is a perfect subset of the continuum, isomorphic to $\mathbf{2}^\omega$, comprises all algebraic numbers and Cohen reals, totally disconnected and first category (meagre) in the Derived set, $\mathbf{2}^{\mathbb{N}_\lambda} \cong \mathbb{R}$. The power set of the second set is not identical to a Suslin line but constructs one, comprising amoeba reals of $\mathbf{2}^{\mathbb{N}_\lambda} \cong \mathbb{R}$.

These results may all seem <u>very new</u>; but in point of fact what has been achieved here is largely a collation of results from other sources that have been very well documented. For example, it seems that the authors of transcendental number theory did not appreciate the relevance of their work to the problem of the continuum. There is also a proliferation of terminology and a tendency to duplicate results.

As an example of this proliferation, let us now observe that the Derived set is identical to the Dedekind-MacNeille completion of $\mathbb{Q}$. The presentation that follows is due to Davey and Priestley [1990] where proofs of any results may be found.

1. Let $x\downarrow$ denote the ideal: $x\downarrow=\{y\in\mathbb{P}:y\le x\}$ where $\mathbb{P}$ is any partially ordered set. Note $\mathbb{P}$ could comprise a single antichain, and this is the case in this context, where is is a skeleton.
2. We derive a lattice, $\mathcal{O}(\mathbb{P})$ by means of the mapping: -

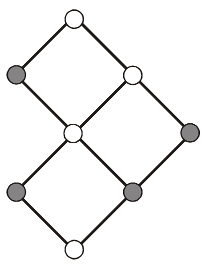
$$\varphi\begin{cases}\mathbb{P}\to\mathcal{O}(\mathbb{P})\\ x\to x\downarrow\end{cases}$$

3. Specifically, let $\mathbb{J}$ be a collection of join-irreducible elements. This means that every element of $\mathbb{J}$ is not the join of any other elements of $\mathbb{J}$. If $x=y\vee z$ where $y,z\in\mathbb{P}$ then $x\notin\mathbb{J}$. Then $\mathcal{O}(\mathbb{J})$ is a derivation from $\mathbb{J}$ comprising the lattice of all ideals generated from $\mathbb{J}$.

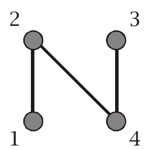
   The meaning of this may not immediately clear, but the map φ gives instructions for the derivation of $\mathcal{O}(\mathbb{J})$ from $\mathbb{J}$ as follows: (a) Given $\mathbb{J}$, find the power set, $\mathbf{P}(\mathbb{J})$. Identify all the join-irredicible elements of $\mathbb{J}$ and take only those ideals that are allowed by relations in $\mathbb{J}$. In $\mathbb{J}$ the maximal element, $\mathbf{1}=\mathbb{J}$, defines a down-set, so every admissible ideal is included in $\mathcal{O}(\mathbb{J})$ with the exception of those deleted by the relations in $\mathbb{J}$. Two examples should further clarify this procedure.

Examples

3.1 Let L be the following lattice: -



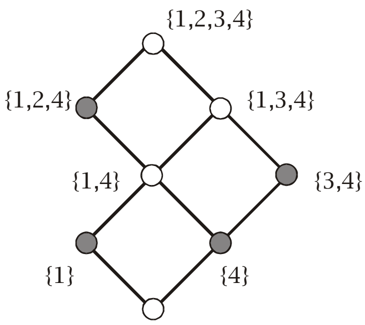
The join irreducible elements (shaded in the above diagram) are given by $\mathbb{J}(L)$: -



The power set is $\mathbf{P}(\{1,2,3,4\})$ but of these elements not all belong to $\mathcal{O}(\mathbb{J})$.

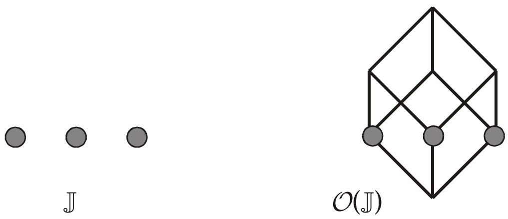
$\{1,2,3,4\}$	✓	$\{1,2\}$	×	$\{1\}$	✓
$\{1,2,3\}$	×	$\{1,3\}$	×	$\{2\}$	×
$\{1,2,4\}$	✓	$\{1,4\}$	✓	$\{3\}$	×
$\{1,3,4\}$	×	$\{2,3\}$	×	$\{4\}$	✓
$\{2,3,4\}$	✓	$\{2,4\}$	×		
		$\{3,4\}$	✓		

This gives the lattice $\mathcal{O}(\mathbb{J})$: -



In this case it is a replica of the original lattice. This is always the case for any distributive lattice and is the meaning of the Birkhoff Representation Theorem.

3.2 An antichain generates the whole power set.



$\mathbb{J}$ $\mathcal{O}(\mathbb{J})$

4. Then the Dedekind-MacNeille completion of $\mathbb{J}$, denoted $\mathrm{DM}(\mathbb{J})$ is isomorphic to $\mathcal{O}(\mathbb{J})$. This illustrates the proliferation of terminology, since the two structures are essentially the same.

5. It is a result that for any Boolean lattice with skeleton $\mathbb{J}$, we have: -
$\mathrm{DM}(\mathbb{J}) \cong \mathcal{O}(\mathbb{J}) \cong \mathbf{P}(\mathbb{J})$.

6. Davey and Priestley [1990] remark:
"It is not difficult to see that every real number $x \in \mathbb{R}$ satisfies $\bigvee_{\mathbb{R}} (x\downarrow)_{\mathbb{Q}} = x = \bigwedge_{\mathbb{R}} (x\uparrow)_{\mathbb{Q}}$ and hence that $\mathbb{Q}$ is both join-dense and meet-dense in $\mathbb{R} \cup \{-\infty, \infty\}$. Consequently $\mathbb{R} \cup \{-\infty, \infty\}$ is (order isomorphic to) the Dedekind-MacNeille completion of $\mathbb{Q}$." (Examples 2.38, p. 44) The Dedekind-MacNeille completion of $\mathbb{Q}$ requires an actually infinite skeleton, which is our $\mathbb{Q}_{\lambda} \sim \mathbb{N}_{\lambda}$; hence $\mathbb{R}$ is here <u>identified</u> with the Derived set, $\mathbf{2}^{\mathbb{N}_{\lambda}}$, and the theory advanced here is just a more explicit exposition of that idea.

7. Furthermore, Davey and Priestley [1990] add: "For any set *X*, the complete lattice $\mathbf{P}(X) \cong \mathrm{DM}(P)$ where $P = \{\{x\} : x \in X\} \cup \{X - \{x\} : x \in X\}$."

   If we substitute ω for *X* in this, and identify the *P* in the above expression with $\mathbb{N}_\lambda$, the skeleton of the continuum, we obtain our decomposition: -

   $\mu = \{\{n\} : x \in \omega\} \cup \{\omega - \{n\} : n \in \omega\}$.

   On completion of the lattice algebra we require an actually infinite skeleton, so we require: -

   $\mathbb{N}_\lambda = \mathbb{N}_A \cup \{\{\lambda\}\} \cup \mathbb{N}_B \cup \mu$.

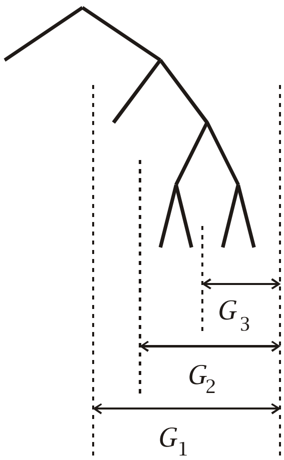
   We observe, again, that here we interpret ω as a representative of the unit interval, and so as having positive measure 1.

Thus, to adapt the words of Davey and Priestley [1990] above, by dropping the caveat "order isomorphic", the arithmetical continuum just <u>is the Derived set</u>; so certainly the Derived set is homeomorphic to the continuum. This is, of course, all subject to the Axiom of Completeness and the Dirichlet pigeonhole principle.

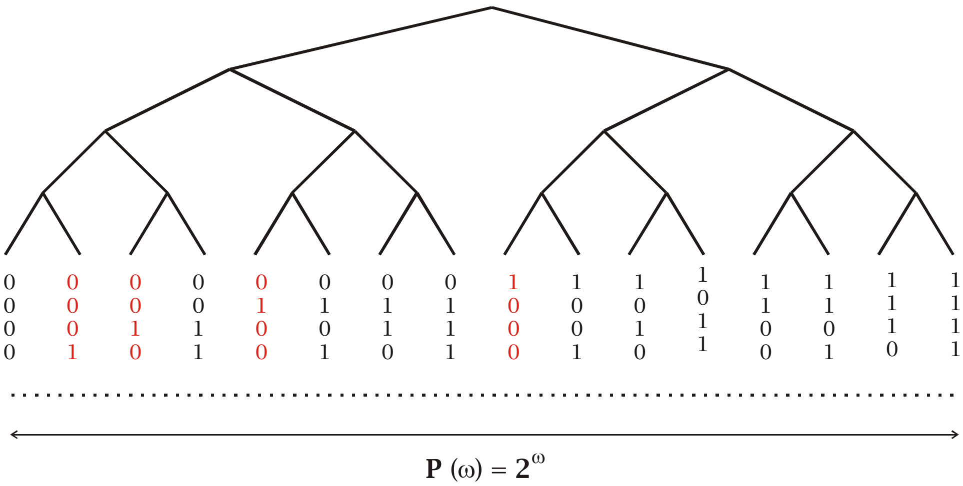
But there is a little more we can say about this homeomorphism in order to make it even more explicit. The Derived set is generated from its skeleton. This skeleton is a decomposition of the unit interval $\mathbb{I} = [0,1]$ in which there are no gaps, and hence continuous and connected. Now we show that the Derived set is the continuous image of $\mathbb{I} = [0,1]$. The derivation is effected by the powerset operation: $\mathbf{2}^{\mathbb{N}_\lambda} = \mathbf{P}(\mathbb{N}_\lambda) \sim \mathbf{P}(\omega) = \mathbf{2}^\omega$. To quote from Davey and Priestley [1990]: "We may regard the maps $L \to J(L)$ and $P \to \mathcal{O}(\mathbb{J})$ as playing a role analogous to that of the logarithm and exponential functions." (Chapter 8, p. 172.) Here the map $L \to J(L)$ is the anti-derivation of the map $P \to \mathcal{O}(\mathbb{J})$ and takes a lattice *L* to its set of join-irreducible elements and the map $P \to \mathcal{O}(\mathbb{J})$ is the power set operation as indicated above: $\mathrm{DM}(\mathbb{J}) \cong \mathcal{O}(\mathbb{J}) \cong \mathbf{P}(\mathbb{J})$. The power set operation, $\exp(n) = \mathbf{2}^n$, in set theory when defined on finite ordinals $n \in \omega$ just is ordinal exponentiation and is a monotonically increasing continuous map. The question is what happens when we reach ω? Here ordinal exponentiation and cardinal exponentiation diverge; however, in our model of the continuum we require the map to progress continuously and smoothly into the neighbourhood of 1 in the unit interval. In ZFC set theory the possibility exists that there is a jump discontinuity at ω, so that $\mathbf{P}(\omega) = \mathbf{2}^\omega \geq \omega_1$. However, the Completeness Axiom closes off this possibility by requiring the continuum to be smoothly generated from the well-orderings of its dense skeleton. Thus, under the exponent map the Derived set is the continuous image of its skeleton.

I must add something more to these remarks in order to clarify in what sense the power set operation, as an exponential function, is continuous. In general topology a function

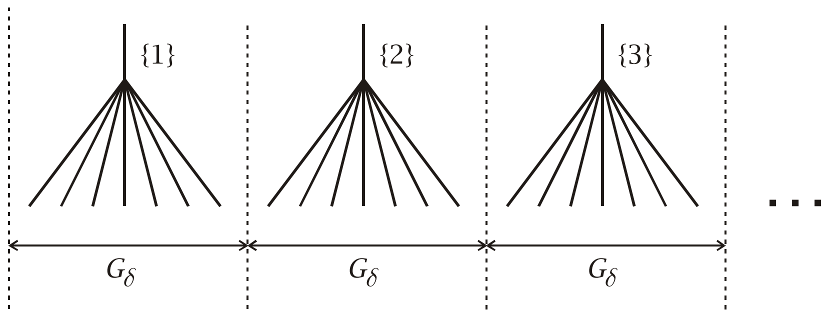
$f : X \to Y$ is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y. So we need to investigate what are the open sets in the skeleton. Considering the lattice of boundaries: -



The boundary points are closed sets belonging to open intervals; these form nested sequences of G_δ sets. This tree extends downwards and generates the boundaries in the potentially infinite skeleton of the continuum. The lattice (Boolean algebra) that is derived from it has exactly the same appearance, only in this Boolean algebra the atoms appear as a set of linearly independent vectors – atoms – a subset of size ω within the entire tree $\mathbf{2}^\omega$.



Rearranging this as a root system, in which each atom of the dense subset is a root of a cluster of reals: -



We see that a set is open in the skeleton iff it is open in the algebra: -

G_δ in $\mathbb{N}_\lambda \Leftrightarrow G_\delta$ in $\mathbf{2}^{\mathbb{N}_\lambda}$.

Thus the derivation is continuous and the Derived Set is homeomorphic to the arithmetical continuum under the Axiom of Completeness.

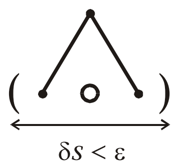
18.3 Binary representations of real numbers

For any real $x \in \mathbb{R}$ in the skeleton we have the standard binary expansion which maps $x \to \mathbf{2}^\omega$. (Levy [2002] Theorem 3.5 p. 236.) This is not technically a bijection because the finite fractions correspond to two binary expansions. For example: -

$$\frac{13}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} = \langle 1,1,0,1,0,0,0, \ldots ,0,0,0, \ldots \rangle = \langle 1,1,0,0,1,1,1, \ldots ,1,1,1, \ldots \rangle$$

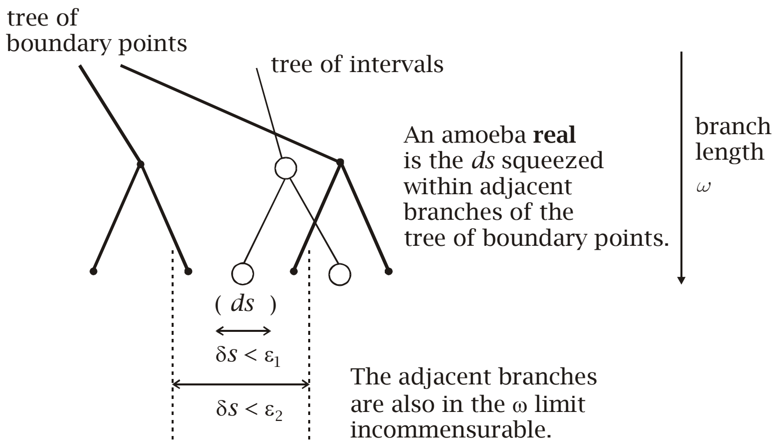
where the lists are actually infinite – that is, an actually infinite list of 0's in the first, and an actually infinite list of 1's in the second. But in the limit, these two lists are incommensurable and are identified as points; they are no longer identifiable in the continuum as distinct branches in the Cantor tree. So we have a bijection between the reals and the sequences in the Cantor tree. We have a continuous bijection between the real line and the Derived set, so they are homeomorphic.

I shall conclude with saying a little more about the binary expansions of real numbers. A Cohen real is identified with the space between two terminal branches of the Cantor tree of the dense set of boundaries: -



The two termini belong to the same terminal node, which at the ωth stage are incommensurable; so the branches are indistinguishable on the continuum, even if they might be distinguishable elsewhere. Hence, a Cohen real has just one binary expansion, and is approximated at any given stage by just one binary approximation.

The situation for amoeba reals is more complex.



Here we see that a amoeba real lies between different branches of the Cantor tree. The interval between these branches is squeezed to an incommensurable measure, here shown as $\delta s < \varepsilon_2$, but it is the essence of a amoeba real to have two approximations. At the final level these are incommensurable, but at every prior stage an amoeba real is represented as an interval. This confirms the fundamental difference between Cohen and amoeba reals, and the distinction in the Mahler classification of transcendental numbers between *U, T* numbers on the one hand, which we here identify with the Cohen reals, and the *S* numbers on the other, which we here identify with the amoeba reals.

References

Aristotle [2012 / c 350 BC]

Physics. Trans. R. P. Hardie and R.K. Gaye. University of Adelaide. Published as an e-book.

Baker, Alan [1975]

Transcendental Number Theory. Cambridge University Press.

Bartoszynski, Tomek and Judah, Haim [1995]

Set Theory on the Structure of the Real Line. A.K. Peters. Wellesley, Massachusetts.

Boolos, George and Jeffrey, Richard [1980]

Computability and Logic. Second edition. Cambridge University Press. Cambridge. (First edition, 1974)

Bourbaki, Nicholas [1989a]

General Topology, Chapters 1 – 4. Springer-Verlag. Berlin. First pub. 1980

Bourbaki, Nicholas [1989b]

General Topology, Chapters 5 – 10. Springer-Verlag. Berlin. First pub. 1980

Burger, Edward B. and Tubbs, Robert [2004]

Making Transcendence Transparent. Springer. New York.

Cohen, Paul J. [1966]

Set Theory and the Continuum Hypothesis. W. A. Benjamin, Inc. New York.

Davey, B. A and Priestley, H.A [1990]

Introduction to Lattices and Order. Cambridge University Press. Cambridge.

Givant, Steven and Halmos, Paul [2009]

Introduction to Boolean Algebras. Springer.

Halbeisen, Lorenz J. [2012]

Combinatorial Set Theory. Springer.

Hardy, G.H. [1908 / 1967]

A Course of Pure Mathematics. Tenth Edition. Cambridge University Press.

Komjáth, Péter and Totik, Vilmos [2000]

Problems and Theorems in Classical Set Theory. Springer. New York.

Kunen, Kenneth [1980]

Set Theory, An Introduction to Independence Proofs. North-Holland. Amsterdam.

Levy, Azriel [2002]

Basic Set Theory. Dover. New York.

Oxtoby, John C. [1980]

Measure and Category. Second Edition. Springer-Verlag. New York.

Potter, Michael [2004]

Set Theory and its Philosophy – A Critical Introduction. Oxford University Press.