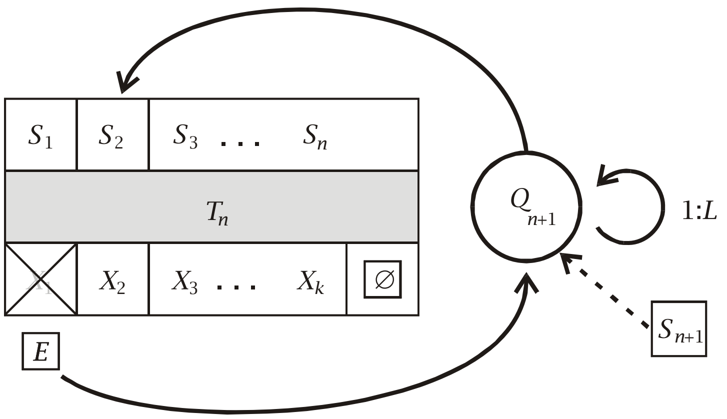
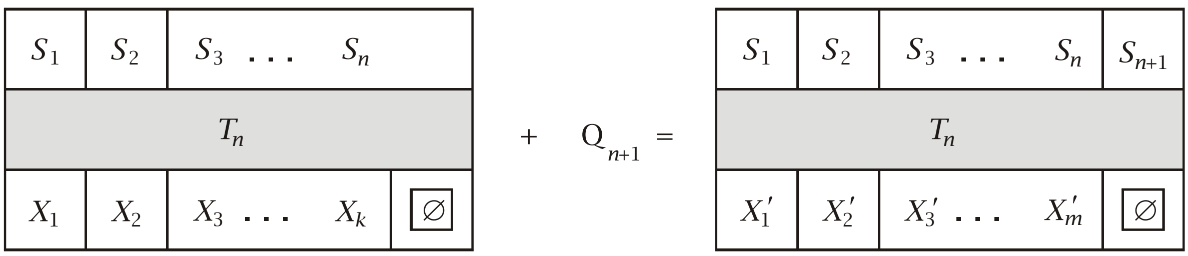
The solution to the halting problem

The problem is to determine the "complete criterion" for halting and non-halting tape configurations for a Turing machine. The diagram that follows illustrates the intuition that a proof by mathematical induction is certain. An example of a T_{n+1} machine is: -



$S_1, S_2, \ldots, S_n$ are input configurations; $Q_1, Q_2, \ldots, Q_n$ are states of machine T_n; $X_1, X_2, \ldots, X_k$ are exit, halting configurations. $\varnothing$ is the internal non-halting loop. In attaching T_n to new state Q_{n+1} we change one or more of the termini in T_n to exits $\boxed{E}$ and direct these towards Q_{n+1}. The induction hypothesis is that the criterion for T_n is completely determined, and the particular step is that from the classification of all 1-state machines that there is a complete criterion for each. The induction step follows from a finite closure property. The problem takes the form



<u>Finite closure property</u>

Let "the problem for ..." be short for "the problem of determining the complete criterion for ...". Then the proof is as follows: Let the problem for T_n be finite. We obtain T_{n+1} by attaching Q_{n+1} to T_n. The problem for Q_{n+1} is finite. Therefore the problem for T_{n+1} is finite. This is a closure property: the property "the problem for T_k is finite" is closed under the addition of finite information.

<u>Remark</u>

The complete criterion for any finite Turing machine can be determined by inductively constructing the machine from a 1-state machine in finite steps as in the above diagrams, and solving at each stage.

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Definitions, determined, complete criterion

Let T_n represent a Turing machine. Then the statement: T_n *is determined* represents the property of T_n that for any given input configuration of finite information of the Turing tape with starting symbol *S* in state Q_i then it is determined:-

1. Whether T_n halts or does not halt when the machine is started with configuration in any of its internal states $Q_i, 1 \le i \le n$.
2. If the machine halts, then the terminal configuration is known.

The information contained in a solution to this problem is called a *complete criterion*.

Complete criterion theorem

A complete criterion may be written for any Turing machine.

Proof

The proof is by complete induction.

Particular step

For $n = 1$. It is evident from the classification of all 1-state machines that there is a complete criterion for each. It is also solved by Boolos and Jeffrey [1998] on p. 35.

Induction step

The problem takes the form

S_1	S_2	S_3 . . . S_n	
T_n			
X_1	X_2	X_3 . . . X_k	$\boxed{\varnothing}$

+ Q_{n+1} =

S_1	S_2	S_3 . . . S_n	S_{n+1}
T_n			
X_1'	X_2'	X_3' . . . X_m'	$\boxed{\varnothing}$

$S_1, S_2, \ldots, S_n$ are input configurations; $Q_1, Q_2, \ldots, Q_n$ are states of machine T_n; $X_1, X_2, \ldots, X_k$ are exit, halting configurations. $\varnothing$ is an internal non-halting loop. In attaching T_n to new state Q_{n+1} we change one or more of the termini in T_n to exits $\boxed{E}$ and direct these towards Q_{n+1}. The induction hypothesis is that the criterion for T_n is completely determined.

Proof of the induction step by the method of exits

The method of exits works by tracing back information from every terminus through the machine $T_{n+1} = T_n + Q_{n+1}$. In doing so we work around each loop in T_{n+1}, recording any 1-loop in it by an asterisk and each longer loop by a bar symbol. [See paper.] These encode the possibility of a finite repetition of a configuration leading to an exit (halting configuration) as well as identifying the infinitely recurring non-halting configurations. The problem is finite since the period of the maximal cycle in T_{n+1} is finite. But if the period of the maximal cycle in T_n is finite then the addition of Q_{n+1} adds a finite number of loops to the maximal cycle; and the resultant maximal cycle and its period remain finite. Therefore, the problem can be solved by the method of exits for T_{n+1}. [There is also a proof by the method of "inputs".]