MELAMPUS

The formal contradiction implied by the assumption that the universalized Gödel's theorem is recursive

1. John Lucas claims that "Gödel's theorem states that in any consistent system which is strong enough to produce simple arithmetic there are formulas which cannot be proved in the system, but <u>which we can see to be true</u>" (my underlining). [Quoted in FRANZÉN 2005 p.117.] He draws the conclusion "that no machine can be a complete or adequate model of the mind, that minds are essentially different from machines." [LUCAS 1961]

2. There is a related argument by Roger Penrose where he denotes Gödel's theorem by the symbol $P_k(k)$: "The procedure that suggests itself is the following. Let us accept that $P_k(k)$, which for the present I shall denote by G_0, is indeed a perfectly valid proposition; so we may simply adjoin it to our system, as an additional axiom. Of course, our new amended system will have its *own* Gödel proposition, say G_1, which again is seen to be a perfectly valid statement about numbers. Accordingly we adjoin G_1 to our system also. This gives a further amended system which will have its own Gödel proposition G_2 (again perfectly valid), and we can then adjoin this, obtaining the next Godel proposition G_3, which we also adjoin, and so on, repeating this process indefinitely. What about the resulting system when we allow ourselves to use the *entire* list $G_0, G_1, G_2, G_3, \ldots$ as additional axioms? Might it be that *this* is complete? Since we now have an unlimited (infinite) system of axioms, it is perhaps not clear that the Gödel procedure will apply. However, this continuing adjoining of Gödel propositions is a perfectly systematic system of axioms and rules of procedure. This system will have its own Gödel proposition, say G_ω, which we can again adjoin, and then form the Godel proposition $G_{\omega+1}$, of the resulting system. Repeating, as above, we obtain a list $G_\omega, G_{\omega+1}, G_{\omega+2}, G_{\omega+3}, \ldots$ of propositions, all perfectly valid statements about natural numbers, and which can all be adjoined to our formal system. ..." [PENROSE 1989, p.142]

3. Penrose appears to accept that the encoding $G_i \to G_{i+1}$ from any G_i in this list to its successor G_{i+1} where *i* ranges over all ordinals whatsoever, finite and transfinite, is algorithmic. He concludes: "This [adjoining of an infinite family of Gödel propositions] requires that our infinite family can be systemized in some algorithmic way. To be sure that such a systematization *correctly* does what it is supposed to do, we shall need to employ *insights* from outside the system – just as we did in order to see that $P_k(k)$ was a true proposition in the first place. It is these insights that cannot be systematized – and, indeed, must lie outside *any* algorithmic action!" [PENROSE 1989, p. 143].

4. Both Lucas and Penrose distinguish two versions of Godel's theorem – the first being a statement, denoted G_0, about a particular sufficiently strong logic, and the second being one about an infinite list of logics, each equipped with a Godel proposition, thus generating an infinite list of Godel propositions: $G_0, G_1, G_2, G_3, \ldots, G_\omega, G_{\omega+1}, G_{\omega+2}, G_{\omega+3}, \ldots$. There is in both arguments an implicit induction, an argument from *any* to *all*. However, neither Lucas nor Penrose, explicitly make that inference.

5. The standard reply to both Lucas and Penrose is summed up by Franzén: "What we *know* on the basis of Gödel's proof of the incompleteness theorem is not that the Gödel sentence *G* for a theory *S* is true, but only the implication, "If *S* is consistent then *G* is true." This implication is provable in *S* itself, so there is nothing in Gödel's proof to show that we know more than can be proved in *S*, so far as arithmetic is concerned." [FRANZÉN 2005, p. 117].

6. Notwithstanding if we do allow the universalization of Gödel's proposition then a formal contradiction is entailed, as I shall now proceed to demonstrate. A logic is defined by a set of axioms and a set of rules of inference. Denote a logic by *K* and a set of axioms by Σ. An inference within *K* then takes the form: $\Sigma \vdash_K \varphi$ where φ is a wff of *K*. The relation of consequence is denoted $\Sigma \vDash_K \varphi$. Godel's proposition can itself be summarized very succinctly; let *Q* denote the proposition:

$Q = \Sigma \nvdash_K Q$	*Q* is defined to the be proposition, "There is not a proof of *Q* in *K* from Σ."

The whole *labour* of the proof of Gödel's proposition is to show that *Q* is a genuine recursive wff of *K*. This is technical, but not required for our purpose. From the above form Gödel's theorem can be easily demonstrated by reductio:

Suppose $\Sigma \vdash_K Q$. Then

$$\Sigma \vdash_K (\Sigma \nvdash_K Q)$$
$$\Sigma \nvdash_K Q$$

This also entails $\Sigma \vDash_K Q$. Hence we obtain:

One-step Gödel theorem

Let *K* be *any* a consistent, sufficiently strong logic. Then

$G_{\text{ONE}} \qquad (\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$

In words, "There exists at a statement *X*, such that there is no proof of *X* in *K* from the axioms Σ, but *X* is a consequence in *K* of Σ." There may be many such statements *X*, but Godel's theorem explicitly constructs one such statement, $Q = \Sigma \nvdash_K Q$. *Q* is an instance of G_{ONE}. Allowing the universalization of this statement, we obtain

Universal Gödel theorem

Let *K* be *any* a consistent, sufficiently strong logic. Then

$G_{\text{UNV}} \qquad (\forall \Sigma)(\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$

In words, "Given a sufficiently strong logic *K,* then for **all** extensions of *K* formed by adjoining new axioms to *K* to form a set of axioms Σ there exists at a statement *X*, such that there is no proof of *X* in *K* from the axioms Σ, but *X* is a consequence in *K* of Σ." This is an argument in the meta-logic about the logic *K*. Denote the meta-logic by Ω and the axioms of the meta-logic by Γ Then we have in the meta-logic a proof of G_{UNV}. That is: $\Gamma \vdash_{\Omega} G_{\text{UNV}}$. The standard formalist reply to any mentalist interpretation of this drawn by Lucas and Penrose is that the proof in the meta-logic is also first-order. (See Franzén, already quoted above.)

7. To show that this formalist rejoinder is invalid, now suppose that the meta-logic, Ω, is itself a consistent, sufficiently strong *first-order* logic. More specifically, that Ω has the same set of inferences as *K* and that the axioms of Ω are such that $\Gamma = \Sigma_j$ for some ordinal *j*. Denote this set of axioms by $\Gamma = \Sigma*$. Then

$\Sigma* \vdash_K G_{\text{UNV}}$	
$\Sigma* \vdash_K (\forall\Sigma)(\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$	Substituting for G_{UNV}
$\Sigma* \vdash_K (\exists X)(\Sigma* \nvdash_K X \text{ and } \Sigma* \vDash_K X)$	Universal instantiation, $\Sigma = \Sigma*$
$\Sigma* \vdash_K (\Sigma* \nvdash_K Q \text{ and } \Sigma* \vDash_K Q)$	Where Q is the specific Godel proposition for $\Sigma*$
$\Sigma* \vdash_K \Sigma* \nvdash_K Q \text{ and } \Sigma* \vdash_K \Sigma* \vDash_K Q$	
$\Sigma* \vdash_K \nvdash_K Q \text{ and } \Sigma* \vdash_K \vDash_K Q$	
$\Sigma* \nvdash_K Q \text{ and } \Sigma* \vdash_K Q$	

So the assumption that the **universal** Gödel's theorem is one of the first-order logics leads to a formal contradiction, albeit that the assumption that the **one-step** Gödel's theorem is first-order does not entail such a contradiction.[1]

8. That this is a valid conclusion can be best affirmed from semantic rather than from syntactic arguments; that is, by examination of the models of first-order logics. But firstly, let me address two prima facie objections.

Objection 1

The argument is relative to a logic *K*. If we assume that *G* is proven in some other logic, say $K*$, then the formal contradiction does not arise. That is, from

$$\Sigma* \vdash_{K*} G_{\text{UNV}}$$

we obtain:

$$\Sigma* \vdash_{K*} \nvdash_K Q \text{ and } \Sigma* \vdash_{K*} \vDash_K Q$$

This is **not** a contradiction, and we cannot proceed further.

Reply 1

K is a sufficiently strong, consistent first-order logic. That is, it is a combination of the *complete* first order predicate logic with the addition of axioms (of set theory) that make it sufficiently strong to express the necessary arithmetical relations for Gödel's theorem

[1] This is exactly analogous to the anti-diagonalisation argument used by Cantor to establish that the cardinality of the reals is strictly larger than the cardinality of the rationals.

to go through. In so far as K is a first-order logic it has rules of inference that make it complete, as demonstrated by the Gödel-Henkin completeness theorem. Therefore, it is not possible to adjoin to K any further rules of inference, or if any additional rule of inference is added, then the deductive power of the logic is not thereby increased: "… that predicate logic is complete means that the rules of reasoning used in predicate logic are sufficient to derive every logical consequence of a set of axioms in a first-order language." [FRANZÉN 2005, p. 27]. In other words, the only way to extend K is by adding axioms, not by adding rules of inference. Furthermore, if we allow ourselves to quantify over *all* sufficiently strong first-order logics, as Gödel's theorem evidently permits, then the formal contradiction does go through. That is,

$$G'_{\mathrm{SECONDORDER}} \qquad (\forall K)(\forall \Sigma)(\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$$

Here we quantify over all (consistent, sufficiently strong) logics, K. However, this statement is second-order since it quantifies over sets of sets, so is not a candidate for a proof in first-order logic. To avoid the possibility of a formalist rejecting the argument on this account, what is required is a first-order statement from which a contradiction can be derived. The statement

$$G_{\mathrm{UNV}} \qquad (\forall \Sigma)(\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$$

is first-order, because we quantify over the sets of axioms Σ rather than over logics K.[2]

<u>Remark</u>

The objection goes to the essence of the problem. The meta-language cannot be a first-order logic. Thus, indeed, there are rules of inference in the meta-language that are **not** rules of any first-order language. So indeed we do have:

$$\Sigma * \vdash_\Omega \nvdash_K Q \text{ and } \Sigma * \vdash_\Omega \vDash_K Q$$

where Ω is the meta-language. Any mathematics sufficient to demonstrate the **universal** Gödel theorem cannot be first-order. This opens up the real question of what the differences between first and second order logic are, and refutes the claim often made by formalists that all mathematics is derivable in ZFC. For an example of this claim: "ZFC is a remarkable first-order theory. All of the results of contemporary mathematics can be expressed and proved within ZFC, with at most a handful of exceptions.[3] Thus it provides the main support for the formalist position regarding the formalizability of mathematics.

[2] It is usual to represent this set of axioms as a union of one set with another; that is, an extension formed by $\Sigma' = \Sigma \cup \varphi$. However, this is not correct. In the relation $\Sigma \vdash_K Q$ we derive Q from potentially **all** the axioms. Σ is a conjunction of each axiom: $\Sigma = \varphi_1 \wedge \varphi_2 \wedge \ldots$, where each φ_i is an axiom; so it is not a disjunction as $\Sigma = \varphi_1 \cup \varphi_2 \cup \ldots$ would imply; in terms of the Boolean representation theorem this corresponds to a lattice *meet* rather than a *join*. So we represent the process of forming extensions to axioms by $\Sigma' = \Sigma + \varphi$ and not by $\Sigma' = \Sigma \cup \varphi$.

[3] This "at most" is disingenuous. It includes the second-order axiom of induction and the second-order axiom of completeness. The former is the foundation of arithmetic and the latter the foundation of analysis. So "at most" is a much larger collection than Wolff's remark implies.

In fact, logicians tend to think of ZFC and mathematics as practically synonymous." [WOLFF 2005, p. 36]. This claim about ZFC is false.

9. Objection 2

The universal Godel theorem

$G_{\mathrm{UNV}} \qquad (\forall\Sigma)(\exists X)(\Sigma \nvDash_K X \text{ and } \Sigma \vDash_K X)$

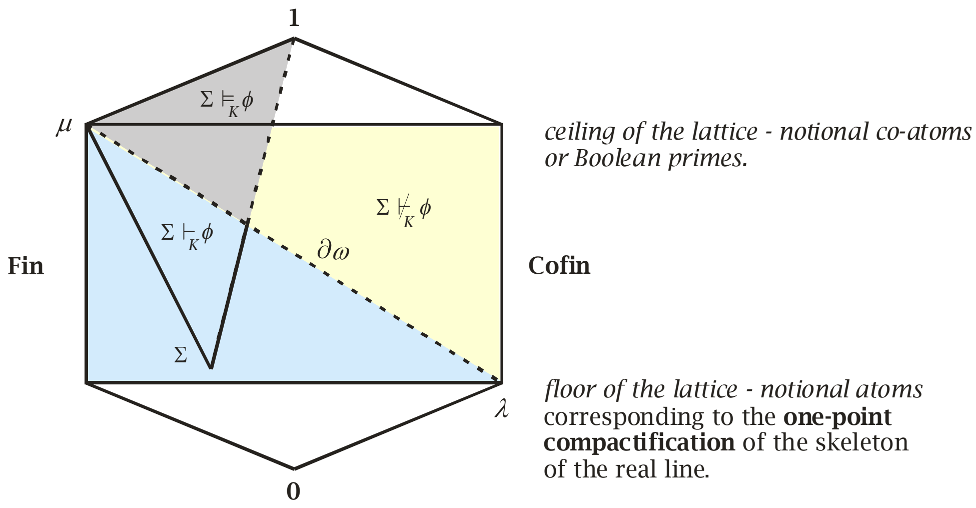
is not admissible in first-order logic. This is because the domain of all sets of axioms over which the quantifier $(\forall\Sigma)\ldots$ ranges is a proper class and not a set.

Reply 2

In deriving the formal contradiction we assume that the meta-language is a copy of the sufficiently-strong first order logic that is a copy of ZFC. It may possibly have new axioms, but no additional rules of inference. Hence, the meta-language is equipped with transfinite induction. The set of all well-formed formulas is recursive, and therefore we can induct over that set. Any well-formed formula may be adjoined to form a set of axioms Σ. That most of these sets will be inconsistent is not a problem, since from an inconsistent statement anything follows, so the proposition $(\exists X)(\Sigma \nvDash_K X \text{ and } \Sigma \vDash_K X)$ follows from Σ even if it is inconsistent as a proposition (for example, $\Sigma = \varphi \wedge \neg\varphi$).

10. The formal contradiction demonstrated here is really only the starting point for an extensive investigation into the *meaning* of the (one-step) Gödel's theorem, $(\exists X)(\Sigma \nvDash_K X \text{ and } \Sigma \vDash_K X)$; this requires a consideration of models. Here I can only indicate the direction in which my own investigation has already proceeded. Set theory was originally devised by Cantor as a model of the continuum, and the continuum is a model of Gödel's theorem. Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\}$ be the one-point (Alexandroff) compactification of the skeleton of the real line, which is standard in the theory of Boolean algebras and lattices – see, for example, [DAVEY & PRIESTLEY 1990][4]. Then the continuum is the power set of this skeleton, which I denote $\mathbf{2}^{\mathbb{N}_\infty}$ and call the "derived set" (since it is derived from $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\}$). Within this set $\Sigma \vDash \varphi$ defines a filter (an "up-set") representing the consequences of Σ, so that $X \in \mathbf{filter}(\Sigma)$ iff $\Sigma \vDash_K X$. The continuum is separated by two mutually disjoint sets: the set of all finite subsets of $\mathbf{2}^{\mathbb{N}_\infty} \sim \mathbf{2}^{\omega}$, denoted **Fin** and the set of all cofinite subsets of $\mathbf{2}^{\mathbb{N}_\infty} \sim \mathbf{2}^{\omega}$, denoted **Cofin**. Between **Fin** and **Cofin** there is a **boundary**. This boundary is inexhaustible and comprises the set of all transfinite numbers. The whole derived set $\mathbf{2}^{\mathbb{N}_\infty}$ is the continuum. All compact proofs $\Sigma \vdash \varphi$ belong either to **Fin** or to **Cofin**. **Fin**, **Cofin** and the **boundary** are a complete partition of the derived set. I represent all of this information in a diagram of the derived set $\mathbf{2}^{\mathbb{N}_\infty}$:

[4] The one-point compactification is a consequence of the Heine-Borel theorem, which is itself equivalent to the Completeness Axiom (say in the form of the Cantor Nested Interval theorem). So the semantic theory I am sketching here is a second-order theory, not a first-order one.



In this diagram the "floor" represents the skeleton of the real line. The symbol $\lambda \equiv \{\mathbb{N}\}$ is a symbol I use to denote the one-point compactification of this skeleton by adding to a potentially infinite partition of the interval $[0,1)$ a neighbourhood of 1. The skeleton, $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\}$, comprises an actually infinite number of points equinumerous to ω, so the resultant derived set is atomic. However, when $\lambda \equiv \{\mathbb{N}\}$ is omitted, the remaining part of the skeleton comprises $\mathbb{N}$ parts, which is potentially infinite and equinumerous to $< \omega$, and the resultant Boolean algebra $\mathbf{2}^{<\omega}$ is non-atomic. The pale blue region represents **Fin**, the region of compact proof paths starting from the set of axioms Σ, which corresponds to a lattice point; that is to a meet of propositions. Outside this region we have $\Sigma \nvdash_K X$. The pale yellow region is **Cofin**. Between **Fin** and **Cofin** is the boundary, $\partial\omega$, which comprises all transcendental numbers of the continuum, constructed as limit points of generic ultra-filters and ideals. The boundary is of cardinality continuum, but **Fin** and **Cofin** together are only of cardinality $\aleph_0$, so the diagram does not represent the differing sizes of the regions – the boundary is much larger than the blue and yellow shaded regions. The region shaded in pale grey, which includes the boundary between it and **Fin**, represents the domain of instances of Gödel's proposition: $\Sigma_0 \nvdash_K X$ and $\Sigma_0 \vDash_K X$, where K is any logic sufficiently strong to express the Gödel proposition. For any given logic (K,Σ_0), G_0 is the statement there is a "point", Q_0 of the boundary that is an instance of the relation $\Sigma_0 \nvdash_K X$ and $\Sigma_0 \vDash_K X$. This point **can** be constructed recursively in a first-order meta-logic. Therefore, it is also recursive to adjoin Q_0 to Σ_0, to create a new set of axioms: $\Sigma_1 = \Sigma_0 + Q_0$. Then G_1 is the statement that there is a new point, Q_1 for Σ_1

constructed as lying on the boundary. The process of adding each Q_i in the chain that Penrose refers to:

$Q_0, Q_1, Q_2, Q_3, \ldots, Q_\omega, Q_{\omega+1}, Q_{\omega+2}, Q_{\omega+3}, \ldots$

with corresponding theorems:

$G_0, G_1, G_2, G_3, \ldots, G_\omega, G_{\omega+1}, G_{\omega+2}, G_{\omega+3}, \ldots$

is also recursive. Nonetheless, we "see" that there is another statement, the Universal Gödel theorem,

$$G_{\mathrm{UNV}} \qquad (\forall\Sigma)(\exists X)(\Sigma \nvdash_K X \text{ and } \Sigma \vDash_K X)$$

referring to the totality of all such chains that cannot be recursive. Suppose that it is recursive and is consequently one of the recursive logics *K.* Then there is a Gödel sentence for this logic, and this leads to a contradiction. So it is not possible through any algorithmic process to adjoin this statement to the domain. It belongs to another logic – a second-order logic, which is complete. The statement G_{UNV} expresses the inexhaustibility of the boundary – the fact that no algorithmic process can generate the boundary.

11. Formalists take the view that any given axiom is recursive. For example, "… every axiom of a system is trivially provable in the system." ." [FRANZÉN 2005, p. 33]. Let *A* be an axiom; then we have $A \vdash A$ trivially. But suppose we attempt to adopt G_{UNV} as an axiom and write $G_{\mathrm{UNV}} \vdash_K G_{\mathrm{UNV}}$. where *K* is a first-order logic. But we have already seen that $\Sigma \vdash_K G_{\mathrm{UNV}}$ leads to a contradiction for all Σ; hence $G_{\mathrm{UNV}} \vdash_K G_{\mathrm{UNV}}$ - also a contradiction. $G_{\mathrm{UNV}} \vdash_K G_{\mathrm{UNV}}$ says that there is a compact proof path in a model of the logic K from G_{UNV} to G_{UNV}. As a proof it would correspond a lattice point *G* - that is, a path of zero length. So what the formal contradiction shows is that G_{UNV} cannot correspond to a lattice point of any lattice whatsoever. The meta-logic in which there is a proof of *G* is *essentially different* from the first order logic whose model is a lattice. The meta-logic does not have a lattice model.

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