The Character of Analytic Logic

Melampus

Contents

1. Analogy
2. Analytic logic
3. Boolean lattices as vector spaces
4. The relationship between analytic logic and its lattice
5. Further observations on analytic logics
6. Notes on self-reference

The Character of Analytic Logic

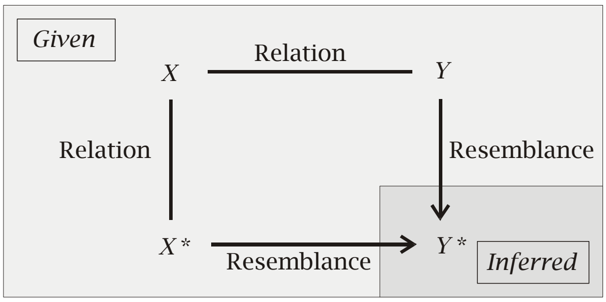
1. Analogy

From Hume's *Dialogues Concerning Natural Religion.*

> That a stone will fall, that fire will burn, that the earth has solidity, we have observed a thousand and a thousand times; and when any new instance of this nature is presented, we draw without hesitation the accustomed inference. The exact similarity of the cases gives us a perfect assurance of a similar event, and a stronger evidence is never desired nor sought after. But wherever you depart, in the least, from the similarity of the cases, you diminish proportionately the evidence, and may at last bring it to a very weak *analogy*, which is confessedly liable to error and uncertainty. After having experienced the circulation of the blood in human creatures, we make no doubt that it takes place in Titius and Maevius; but from its circulation in frogs and fishes it is only a presumption, though a strong one, from analogy that it takes place in men and other animals.[1]

1.1 Analogy

An argument by *analogy* is an inference from something given to something not given on the basis of a double resemblance derived from two relations.



Here *relation* and *resemblance* are primitive notions.

From Hume's example, let X = man, X^* = circulation, Y = frog, Y^* = circulation. This is an inexact analogy in which there are two relations between which we perceive an imperfect similitude.

Exact resemblance

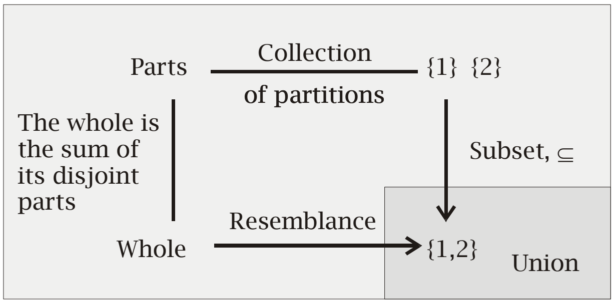
Let R be a relation between X and Y and R^* be a relation between X^* and Y^*. Then the resemblance between R and R^* is said to be *exact* if R and R^* are instances of the same relation: XRY if and only if $X^*R^*Y^*$. X^* and Y^* are *substitution instances* of X and Y and conversely.

[1] Hume [1779] Part 2.

Exact analogy

An *exact analogy* occurs when both resemblances in the analogy are exact.

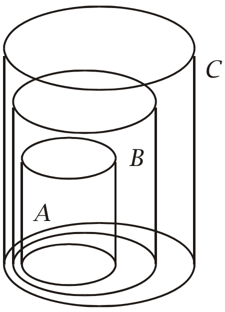
There is an exact analogy in set theory between the subset relation and containment.



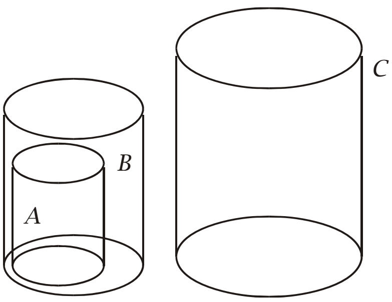
In set theory we talk of *subspace*, *subset*, *empty set*, *domain*, *point set*, and so forth.

2. Analytic logic

The following illustrate how inferences may be derived from an *analogy* with extensional containment in space.



If *A* is contained in *B* and *B* in *C* then *A* must be contained in *C*.



If *A* is contained in *B* and *B* is wholly separate from *C*, then the *A* cannot be contained in *C*. To say that *A* is contained in *C* is a *contradiction*.

Analytic logic (+)

Analytic logic is the science of inferences that originates in an exact analogy with the spatial relation of part to whole.

The propositional calculus is an analytic logic. For example, the tautology, $\vdash (p \wedge q) \supset (p \vee q)$ concerns two propositions, *p* and *q*, each of which can be negated. Let there be a partition of space representing information to which these four propositions apply: p, $\neg p$, q, $\neg q$ subject to combinations of conjunction and disjunction. The space has four disjoint parts with labels:

$\{1\}$ $p \wedge q$ $\{2\}$ $p \wedge \neg q$ $\{3\}$ $\neg p \wedge q$ $\{4\}$ $\neg p \wedge \neg q$

$p \wedge \neg q$ $\{2\}$	$\neg p \wedge \neg q$ $\{4\}$
$p \wedge q$ $\{1\}$	$\neg p \wedge q$ $\{3\}$

The proposition $p \vee q$ corresponds to $\{1,2,3\}$. Since $\{1\} \subset \{1,2,3\}$ if $p \wedge q$ is true then $p \vee q$ must be true; this gives $(p \wedge q) \vdash (p \vee q)$ (From $(p \wedge q)$ infer $p \vee q$.) The tautology $\vdash (p \wedge q) \supset (p \vee q)$ follows. The 16 permutations of $\{1\},\{2\},\{3\},\{4\}$ correspond to 16 partitions of the space represented by $\mathbf{1} = \{1,2,3,4\}$. The collection of all possible unions of $\{1\},\{2\},\{3\},\{4\}$ comprises a *lattice*.

Geometric definition of a lattice

A poset L is called a *lattice* if for every $x, y \in L$, $\sup(x,y) \in L$ and $\inf(x,y) \in L$. Let $x \vee y = \sup(x,y) \in L$ and $x \wedge y = \inf(x,y) \in L$. The element $x \vee y$ is called the "join" of x and y and the element $x \wedge y$ is called the "meet" of x and y.

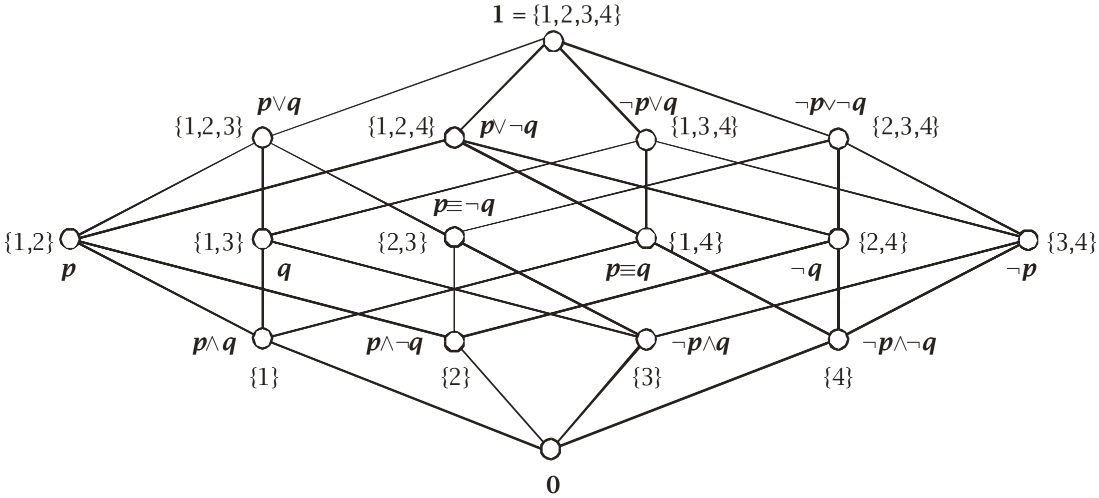
Largest and smallest elements

1 (bold typeface) denotes the largest element in the lattice – the join of all the elements.
0 denotes the smallest element of the lattice – the meet of all the atoms.[2]

The *model* (up to isomorphism) of the propositional logic of *p* and *q* is a Boolean (complemented, distributive[3]) lattice.

[2] Not all lattices have a **1**. *Complete* lattices do. All finite Boolean lattices are complete.

[3] A lattice is distributive iff the identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold in it.



Boolean lattice of two propositions

This lattice shall be denoted, $\mathbf{2}^4 = \mathbf{2}^{2^2}$, where $\mathbf{2} \equiv \{0,1\}$ and

$$\mathbf{2}^4 \equiv \mathbf{2}\times\mathbf{2}\times\mathbf{2}\times\mathbf{2} \cong \{0,1\}^4 \cong \{0, p\wedge q\}\times\{0, p\wedge q'\}\times\{0, p'\wedge q\}\times\{0, p'\wedge q'\}$$

(Cartesian product) Here $p' = \mathbf{1} - p$ denotes the complement of *p*.

Boolean lattice / algebra

A *Boolean lattice*, also called a *Boolean algebra*, is any structure $\langle B, \vee, \wedge, ', \mathbf{0}, \mathbf{1}\rangle$, subject to the axioms:

$(B1)$ $\langle B, \wedge, \vee\rangle$ is a distributive lattice

$(B2)$ $p\wedge\mathbf{0} = \mathbf{0}$ $p\vee\mathbf{1} = \mathbf{1}$

$(B3)$ $p\wedge p' = \mathbf{0}$ $p\vee p' = \mathbf{1}$

$\mathbf{2}^4$ is a paradigm: any general property of $\mathbf{2}^4$ is inherited by all finite Boolean algebras.

3. Boolean Lattices as Vector spaces

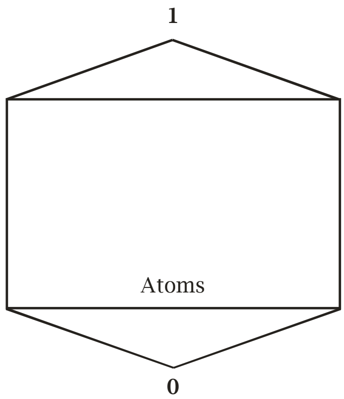
All Boolean lattices (finite or infinite) are vector spaces. $\mathbf{2} \equiv \{\mathbf{0},\mathbf{1}\}$ is a field and $\mathbf{2}^4$ is a vector space over **2** with dimension $\left[\mathbf{2}^4 : \mathbf{2}\right] = 4$[4]. Since $\mathbf{2}^4$ is a vector space, there is a basis for it. For example, the atoms: -

[4] An alternative description of a finite Boolean lattice that makes this transparent is called a Boolean ring $(B, +, \cdot, \mathbf{0}, \mathbf{1})$. The expression $\left[\mathbf{2}^4 : \mathbf{2}\right] = 4$ is read, "The dimension of vector space $\mathbf{2}^4$ over field **2** is 4."

$$\alpha_1 = p \wedge q \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \alpha_2 = p \wedge q' \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \alpha_3 = p \wedge q' \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \alpha_4 = p' \wedge q' \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

A set of *atoms* for a finite Boolean lattice is a finite set of linearly independent vectors that forms a complete basis for it. A lattice is *complete* if every join of atoms exists.

An *atom* cannot be further subdivided: if $\alpha_i,\ \alpha_j,\ i \neq j$ are distinct atoms then their meet denotes the null space, $\alpha_i \wedge \alpha_j = \varnothing$. They are disjoint. It is a result that every finite Boolean lattice has a set of atoms that make it complete; all finite Boolean lattices are *atomic*, meaning that they have a set of atoms.



Schematic diagram of a finite Boolean algebra (+)

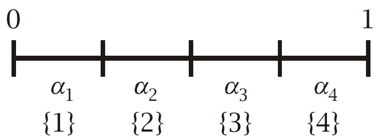
There are also correspondences between the algebraic operations $\wedge,\ \vee,\ '$ and the set-theoretic operations of $\cap,\ \cup,\ '$.

$$\wedge \leftrightarrow \cap \qquad \vee \leftrightarrow \cup \qquad ' \leftrightarrow '$$

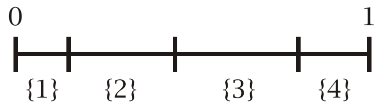
Example

$$\{1,2,3\} \cap \{1,3,4\} = \{1,3\} \qquad \leftrightarrow \qquad (p \vee q) \wedge (\neg p \vee q) \equiv q$$

The partition of the underlying space turns it into a discrete topological space, and the atoms also comprise a topological basis for it. By means of space filling curves any compact (closed, bounded) space may be mapped onto $\mathbf{1} = [0,1] \cong -\infty \cup \mathbb{R} \cup +\infty$ so the primitive structure is a *partition of the continuum* as the *extended real line*, $\mathbb{R}^+ = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, represented most conveniently by $\mathbf{1} = [0,1]$. We have the heuristic diagram: -

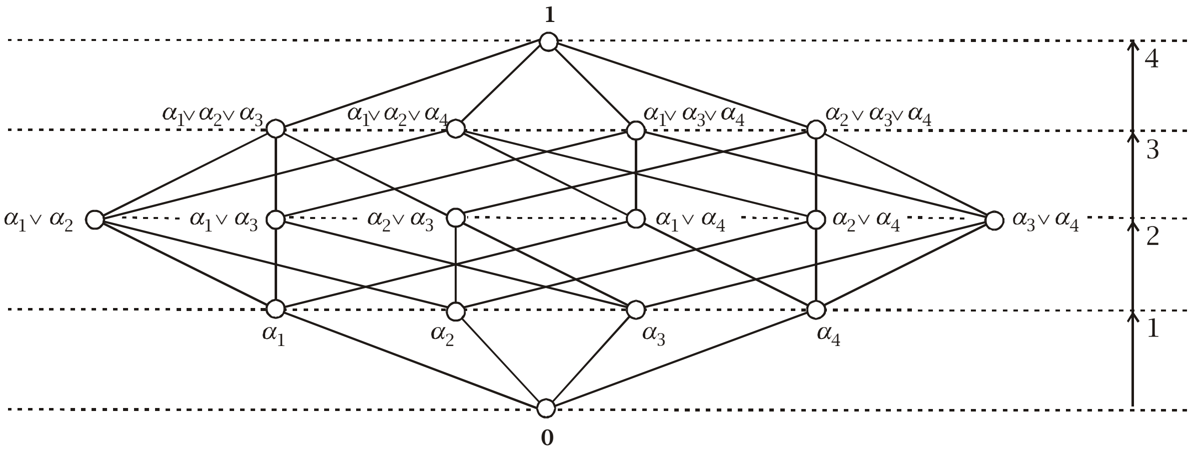


The information contained in the first subinterval here is assigned to the atom α_1. This assignment is extrinsic to the process of generating a lattice. The following is also a possible assignment:



That is, the measure of the interval (or the information it contains) is extrinsic to the lattice.

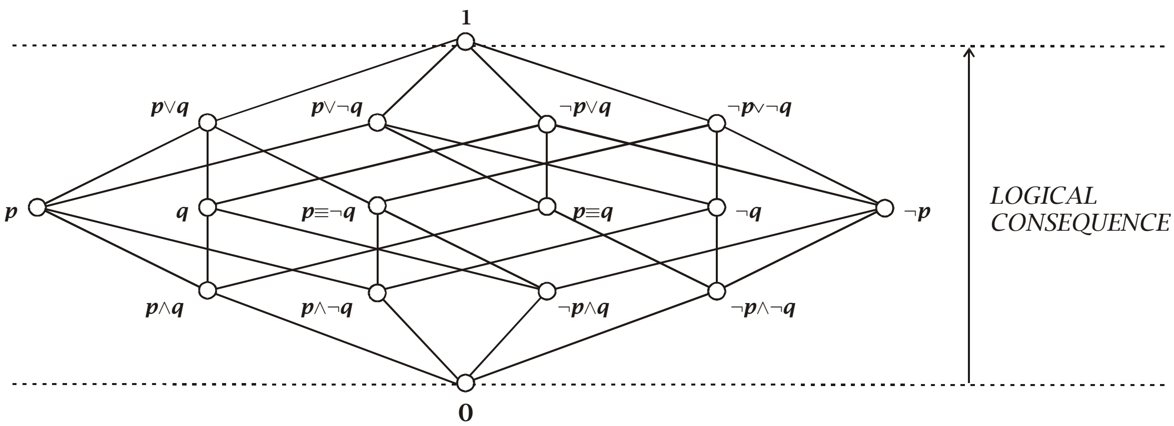
$\mathbf{2}^4$ is also a metric space with an intrinsic unit of measure, the "height" of the lattice point above **0**.



Height is also called the *radial distance from* **0**. For example the height of p in $\mathbf{2}^4$ is 2 (units) since $\mathbf{0} < p \wedge q < p$ is a shortest chain in $\mathbf{2}^4$ joining **0** to p of length 2.

4. The relationship between analytic logic and its lattice

To create analytical logic we must (*a*) add an *extrinsic* sense of direction to the lattice – an *up* and a *down* making it into a graph with directed edges; (*b*) map propositions to lattice points or equivalently assign information to the atoms: -



Consequence is denoted by $\vDash$; in $\gamma \vDash \psi$ γ is called the *premise* and ψ the *conclusion*. This relation is clarified by the concept of *filter* or *up-set*[5]; filters are identified by tracing lines in the lattice diagram upwards. Then every proposition ψ corresponding to a lattice point that lies in the filter corresponding to a lattice point γ is a consequence of γ.

Example

Let $p \wedge q$ correspond in $\mathbf{2}^4$ to the lattice point α_1 and the partition $\{1\}$. The filter of $p \wedge q$ comprises every lattice point where $\{1\}$ is a subset.

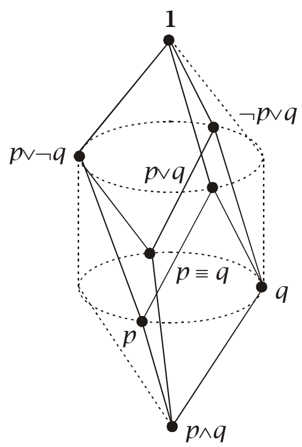
Filter of $\{1\}$

$\{1\} \leftrightarrow p \wedge q$

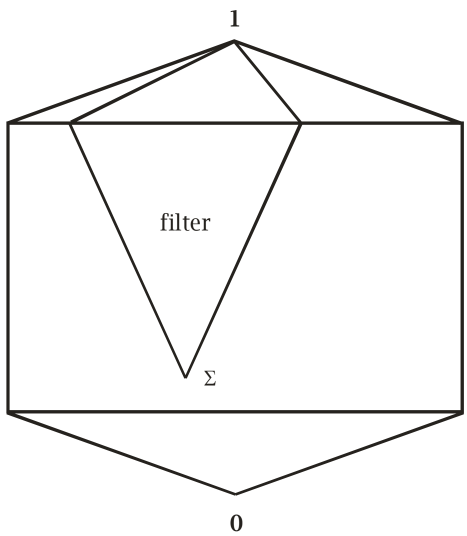
$\{1,2\} \leftrightarrow p$ $\{1,3\} \leftrightarrow q$ $\{1,4\} \leftrightarrow p \equiv q$

$\{1,2,3\} \leftrightarrow p \vee q$ $\{1,2,4\} \leftrightarrow p \vee \neg q$ $\{1,3,4\} \leftrightarrow \neg p \vee q$

$\mathbf{1}$



From the filter: $p \wedge q \vDash p$, $p \wedge q \vDash p \vee \neg q$ and so on.



Schematic diagram of a filter within a Boolean algebra (+)

The concept of *deduction*, symbolised by $\vdash$, is associated with a system of rules of inference each of which allows one to navigate upwards along a path contained in a filter. If we have atoms we only need *one* rule of derivation: -

[5] Davey and Priestley (Davey and Priestley [1990] p. 13

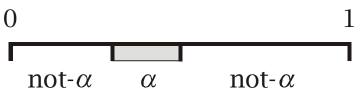
Rule for the introduction of disjunction

Let γ, ψ be any two distinct propositions corresponding to lattice points. Then

$$\frac{\vdash \gamma}{\vdash \gamma \vee \psi}$$

means that from γ we may infer the proposition $\gamma \vee \psi$. This is also written, $\gamma \vdash \gamma \vee \psi$.

The atoms in a finite Boolean lattice conceptually do not exclude negation. An atom, α, marks off a segment of the space $[0,1]$, and partitions it into a part that contingently affirms α and a part that affirms its complement α' which is denoted by $\vdash \neg\alpha$.



The transformation from a system of generators, $\{p, \neg p, q, \neg q\}$, to a system of atoms renders transparent the underlying simplicity and constructible nature of the lattice. For example, in $\mathbf{2}^4$ the generators are $\{p, \neg p, q, \neg q\}$; this is not a linearly independent set, but the atoms are:

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{p \wedge q, p \wedge q', p' \wedge q, p' \wedge q'\}$.

To build a formal system that embodies some rules of inference of natural language under the influence of the spatial analogy (such are the systems of Gentzen[6]) we need negation and conjunction symbols, so we include $\neg$ and $\wedge$ in the logic as well as $\vee$. The construction, *if … then* expresses the ability to attach ideas to one another and "flow" through them; so, we add $\supset$ to the formal language, and also, equivalence $\equiv$.

Principle of dilution

All deductive inference in analytic logic proceeds only by dilution of content; we can never reach a statement of generality greater than that already encompassed by the premises.

Example

The statement $p \supset q,\ p \vdash q$ is equivalent to $p \wedge q \vdash q$, which is valid because q lies in the filter generated by $p \wedge q$. Likewise, in a general Boolean algebra, if $p \wedge q$ represents a disjunction of atoms, $p \wedge q \equiv \alpha_{i_1} \vee \alpha_{i_2} \vee \ldots \vee \alpha_{i_n}$ then q is a *dilution* of this statement by the addition of further atoms to the disjunction: $q \equiv \left(\alpha_{i_1} \vee \alpha_{i_2} \vee \ldots \vee \alpha_{i_m}\right) \vee \left(\beta_{j_1} \vee \beta_{j_2} \vee \ldots \vee \beta_{j_n}\right)$.

There is no single name of a given lattice point: $\neg p \vee q$ and $p \supset q$ are names of the same lattice point. The collection of all names of a lattice point is an *equivalence class*; where we write $\neg p \vee q$ we should write $[\neg p \vee q]$ to denote the equivalence class of which $\neg p \vee q$ is its representative and $p \supset q$ is another

[6] Gentzen produced systems of "natural deduction" and "sequent calculi" that were ostensibly modelled on natural language inferences found in mathematical discourse. See Gentzen [1969] and Ungar [1945].

member. The Boolean algebra generated by these classes is called the *Tarski-Lindenbauum algebra*[7]; it is also known as the algebra of statement bundles.[8]

The canonical name of any tautology is $\mathbf{1}$; we have $\vdash \mathbf{1}$. Members of the equivalence class $[\mathbf{1}]$ are *names* of $\mathbf{1}$. The multiplicity of these different names conceals the dilution principle. Take, for instance, the tautology $\vdash (p \wedge (p \supset q)) \supset q$, which does not *look* like a dilution. However,

$$
\begin{aligned}
&\vdash (p \wedge (p \supset q)) \supset q \\
&\vdash (p \wedge (\neg p \vee q)) \supset q \\
&\vdash \neg(p \wedge (\neg p \vee q)) \vee q \\
&\vdash \neg((p \wedge \neg p) \vee (p \wedge q)) \vee q \\
&\vdash \neg(p \wedge \neg p) \vee \neg(p \wedge q) \vee q \\
&\vdash \neg p \vee p \vee \neg p \vee \neg q \vee q \\
&\vdash \mathbf{1}
\end{aligned}
$$

Similarly, $\mathbf{1} \vdash (p \wedge (p \supset q)) \supset q$. So $\vdash (p \wedge (p \supset q)) \supset q$ is just a diluted form of the law of excluded middle.

4.3 Theorem, canonical representation of names of 1 (+)

Each name of $\mathbf{1}$ has a canonical representation of the form

$\mathbf{1} \# \mathbf{1} \# \ldots \# \mathbf{1}$ or $\mathbf{1} \# \mathbf{1} \# \ldots \# \mathbf{1} \vee \phi$

where $\# = \wedge$ or $\vee$ and $\phi = p_1 \vee p_2 \vee \ldots \vee p_n$, and each p_i is a contingent proposition. In the expression $\mathbf{1} \# \mathbf{1} \# \ldots \# \mathbf{1}$ each $\mathbf{1}$ is a separate name for the law of excluded middle. For example, $\mathbf{1} \vee \mathbf{1} = (p \vee \neg p) \vee (q \vee \neg q)$. We can allow duplication of the same irreducible proposition; for example, $\mathbf{1} \vee \mathbf{1} = (p \vee \neg p) \vee (p \vee \neg p)$.

Proof

The absorption laws for a Boolean algebra give: -

$\phi \vee \mathbf{1} \equiv \mathbf{1}$ $\quad\quad$ $\mathbf{1} \vee \mathbf{1} \equiv \mathbf{1}$ $\quad\quad$ $\mathbf{1} \wedge \mathbf{1} \equiv \mathbf{1}$

Eliminate material implication in favour of joins and meets; i.e $p \supset q \equiv_{df} \neg p \vee q$.

Let α be a wff such that $\vdash \alpha$. Suppose $\alpha \not\equiv (\mathbf{1} \# \mathbf{1} \# \ldots \# \mathbf{1} \vee \phi)$. By a tautology, $\alpha \equiv \neg(\mathbf{1} \# \mathbf{1} \# \ldots \# \mathbf{1} \vee \phi)$. Then: -

---

[7] Mendelson [1979], p. 43

[8] Any set of sentences is not closed under the operations of conjunction and disjunction and therefore does not form a Boolean algebra. (Mendelson Elliott [1970] p.160 et seq.) The conjunctions $p \wedge q$ and $q \wedge p$ are not identical. Therefore, they do not define unique joins in a lattice. So the set of sentences, *per se*, is not a Boolean algebra. This may be a surprising result, since it is natural to think of the set of sentences under the logical operations of conjunction, disjunction and so forth as a Boolean algebra. This difficulty is circumvented by the following definition of equivalence classes on the set of sentences: Define the equivalence class of the sentence p by $[p] = \{q : q \equiv p\}$. That is, as the set of sentences logically equivalent to p. These equivalence classes are also called *statement bundles.* With this definition we may obtain the result that the set of statement bundles is a Boolean algebra.

$$\begin{array}{ll} & \vdash \alpha \\ \Rightarrow & \vdash \neg(\mathbf{1}\#\mathbf{1}\# \dots \#\mathbf{1} \vee \phi) \\ \Rightarrow & \vdash \neg(\mathbf{1} \vee \phi) \\ \Rightarrow & \vdash \neg\mathbf{1} \wedge \neg\phi \end{array}$$

Since we always have $\vdash \mathbf{1}$, this is a contradiction.

5. Further observations on analytic logics

<u>The hierarchy of logics</u>

The properties of $\mathbf{2}^4$ are inherited by all finite Boolean algebras. This is the content of the finite Boolean representation theorem.

5.1 Finite Boolean representation theorem

Every finite Boolean algebra is isomorphic to a field of sets.

<u>Informal proof (+)</u>

Using the notation of algebraic field extensions, we have $[\mathbf{2}:\mathbf{2}]=1$, where $\mathbf{2}=\{0,1\}$ is the Boolean algebra of two elements. Every Boolean algebra $\mathbf{2}^{k+1}$ is a vector space over **2**, and $[\mathbf{2}^{k+1}:\mathbf{2}]=2\times[\mathbf{2}^k:\mathbf{2}]$, since $\mathbf{2}^{k+1}=\mathbf{2}\times\mathbf{2}\times \dots \times\mathbf{2}\ (k+1 \text{ times})$. By induction, for all $n\in\mathbb{N}$, $[\mathbf{2}^n:\mathbf{1}]=2^n$. Each $\mathbf{2}^n$ has a unique basis of vectors that may be placed into one-one correspondence with the singleton sets of elements of some set A where $|A|=n$. We also have $|\mathbf{2}^n|=2^n$.[9]

The lattice $\mathbf{2}^4$ is a paradigm. By the above inductive proof the subject of our enquiry is the class of all finite (Boolean) lattices: $\{B : B\cong\mathbf{2}^{2^n}, n\in\mathbb{N}\}$, and there is a hierarchy of analytic logics: -

2	$\{\mathbf{0},\mathbf{1}\}$	Logic of $0\equiv$ contracdiction; $1\equiv$ necessary truth.
$\mathbf{2}^{2^1}$	$\{\mathbf{0},p,\neg p,\mathbf{1}\}$	Logic of one contingent proposition p and its negation.
$\mathbf{2}^{2^2}$	$\{\mathbf{0},p,\neg p,q,\neg q,\mathbf{1}\}$	Logic of two contingent propositions.
...	...	...
$\mathbf{2}^{2^n}$	$\{\mathbf{0},\dots,\mathbf{1}\}$	Logic of n contingent propositions.
...	...	...

This set is not closed and non-atomic. The hierarchy may be extended to infinite lattices. To demonstrate that the predicate calculus is an analytic logic, we would have to consider infinite lattices.

[9] A more rigorous and formal based on Mendelson [1979] is available. This informal proof exposes the induction involved in the proof. Also note that in this proof the bold type **2** denotes the Boolean algebra of 2 elements; whereas the normal type 2 denotes the number 2.

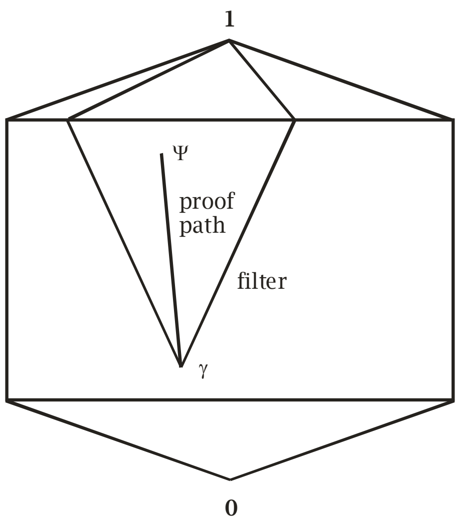
Soundness and completeness

Proof path (+)
Let $A \to B \to C \to \ldots$ be a chain of deductions in an analytic logic, where each consequent is a valid inference from its antecedent. Define the chain of deductions $A \to B \to C \to \ldots$ as a *proof path.*

Let γ, ψ be propositions corresponding to distinct lattice points; then $\gamma \vdash \psi$ means that there is a proof path in the lattice from γ to ψ. If $\gamma \vDash \psi$ then *there must always be a chain of lattice points* from γ to ψ for that is also precisely what $\gamma \vDash \psi$ (consequence) means; $\gamma \vdash \psi$ represents a *finite demonstrable proof* based on rules of inference.[10]

Soundness
An analytic logic is sound if $\gamma \vdash \psi$ then $\gamma \vDash \psi$.

This means at any proof path from a proposition γ corresponding to a lattice point must lie in the filter generated by γ. It is as if a "force" constrained the proof path to stay within the filter.



A proof path is a chain contained within a filter. (+)
The filter represents $\gamma \vDash \psi$ *; the path is* $\gamma \vdash \psi$ *.*

[10] About this aspect Weyl wrote: "As I see it, mathematics owes its greatness precisely to the fact that in nearly all its theorems what is essentially *infinite* is given a finite resolution. But this "infinitude" of the mathematical problems springs from the very foundation of mathematics – namely, *the infinite sequence of the natural numbers and the concept of existence relevant to it.* "Fermat's last theorem," for example, is intrinsically meaningful and either true or false. But I cannot rule on its truth or falsity by employing a systematic procedure for sequentially inserting all numbers in both sides of Fermat's equation. Even though, viewed in this light, this task is infinite, it will be reduced to a finite one by the mathematical proof (which, of course, in this notorious case, still eludes us.)." (Weyl [1994 / 1918] p.49). In mathematical discourse the fundamental reason for distinguishing $\gamma \vDash \psi$ from $\gamma \vdash \psi$ is precisely because human reason seeks *finite proofs* of statements that are essentially *infinite* in conception. Up to now the only lattices we have considered have been finite.

Complete logic
An analytic logic is *complete*, if $\gamma \vDash \psi$ then $\gamma \vdash \psi$.

Completeness means that *if there exists a lattice path then a (finite) proof path can be given for it.*

<u>Intuitionism</u>
By intuitionism here I refer to the formal theory of infinite valued logics wherein the law of excluded middle does not apply. The underlying model of any intuitionist logic is a non-complete distributive lattice. We have the following theorem: -

Theorem
Every distributive lattice can be embedded in a complete Boolean algebra.[11]

So intuitionistic logic (in the sense defined here) is also an analytic logic.

6. Notes on self-reference

In analytic logic the proposition $1 = 0$ represents an identification of the interval $[0,1]$ will the null space. Analytic logic derives its consistency from the partition of the interval $[0,1]$ into disjoint segments, atoms, such that no two atoms can contingently coexist in the sense of a lattice meet (conjunction). (Atoms are disjoint.) The consistency of any analytical logic derives from the relation of part to whole.

Principle of geometric consistency
A space cannot be a proper subspace of itself.

Conjecture 6.1
In an analytic logic self-reference is contradictory and is outlawed as meaningless. The discourse of self-reference has no model within analytic logic.

Proposition 6.2
There exists a well-ordered hierarchy of Boolean lattice extensions, whose finite portion is the hierarchy of all finite Boolean algebras: -

$\mathbf{2}$	$\{\mathbf{0},\mathbf{1}\}$	Logic of $0 \equiv$ contracdiction; $1 \equiv$ necessary truth.
$\mathbf{2}^{2^1}$	$\{\mathbf{0},p,\neg p,\mathbf{1}\}$	Logic of one contingent proposition p and its negation.
$\mathbf{2}^{2^2}$	$\{\mathbf{0},p,\neg p,q,\neg q,\mathbf{1}\}$	Logic of two contingent propositions.
...	...	...
$\mathbf{2}^{2^n}$	$\{\mathbf{0}, \ldots, \mathbf{1}\}$	Logic of n contingent propositions.
...	...	...

[11] For proof, see Crawley and Dilworth [1973] p.89. Also Birkhoff [1940]

Informal proof

This hierarchy is well ordered being based on the partition (skeleton) of the real line $\mathbb{R}_\infty \cong [0,1]$ by an ordinal. Partition the real line into α segments, where α is an ordinal, finite or infinite. Generate a Boolean lattice upon this partition. Then the hierarchy of Boolean lattices is also well-ordered.

Proposition 6.3

First order-predicate logic is an analytic logic whose model is any member of the proper class of Boolean lattices in the hierarchy of lattice extensions given in the preceding proposition.

Proposition 6.4

A hierarchy of Boolean lattices may be generated by repeated iterations in a sequence of meta-languages.

Names

Every element of a Boolean lattice has a number of canonical names. For example, if α_1, α_2 are canonical names for atoms in a lattice, then there is a lattice point with canonical name $\alpha_1 \vee \alpha_2$; but in logic, denoting α_1, α_2 by $p \wedge q$ and $p \wedge \neg q$ respectively, where *p, q* denote propositions, then $\alpha_1 \vee \alpha_2$ also has canonical name p. Such canonical names do not lead to formal contradiction.

Quotation marks

However, in natural language we are also allowed to form names by the use of quotation marks. Names so formed may be completely arbitrary; for example, I may let '*x*' be a name for *p*. Tarski's rule for the use of quotation marks, $'p'$ iff p, permits in principle a violation of the principle of geometric consistency and so allows for the derivation of a contradiction: $\mathbf{0} \equiv \mathbf{1}$, where **0** and **1** denote the least and greatest elements of a lattice respectively.

The fundamental reason why this is so, is as follows. If I assert $\vdash p$ then *p* is affirmed contingently, or in other words *p* is an assumption. Only $\vdash \mathbf{1}$ may be asserted without restriction, since **1** is the canonical name of the lattice point that is the join of all the atoms, and in logic has representatives that are all the tautologies or self-evident truths. Also we have $\nvdash \mathbf{0}$. But the rule $'p'$ iff p allows one to carry over the structure inherent in *p* from the citation to the canonical name, thus: $\vdash 'p' \Rightarrow \vdash \ p$. In this way the name of a lattice point *p* may be asserted as if it were an atom of a new lattice, and this in effect attempts to embed a space as a proper subset of itself, or of another disjoint space.

In fact, unrestricted quotation marks allow one to contingently affirm **0**, which leads automatically to contradiction, as Tarski's famous papers demonstrate. Consider:

$(\forall p)(c = 'p' \supset \neg p)$
$'c'$ is a name of $(\forall p)(c = 'p' \supset \neg p)$ 'This sentence is false'
$\vdash 'c'$
$\vdash (\forall p)(c = 'p' \supset \neg p)$
$\vdash 'c' = '\mathbf{0}'$
$\vdash c$ $'c'$ iff c
$\vdash \mathbf{0}$
$\vdash \mathbf{0} \wedge \vdash \neg \mathbf{0}$

The only way to prevent this is not to permit unrestricted quotation: any quotation marks introduced must preserve the underlying structure of the lattice. This can only be achieved if the quotation marks have the effect of embedding the given lattice into a larger lattice, which is a lattice extension. This is equivalent to a refinement of the partition of the information that forms the skeleton of the lattice from which the (Boolean) lattice is derived.

Any embedding of one lattice in another is a process belonging not to a given lattice and its object language, but to the meta-language. (Of course, it may also be that the meta-language is itself a first-order logic with correspondent Boolean lattice, so one thereby does not break out of the class of all Boolean algebras; but this is another issue.)

In other words, the restriction of quotation whereby the lattice structure be preserved as an embedding of one lattice within another is the precise solution to this problem proposed by Tarski in his famous papers who defines a sequence of lattice extensions each acting as the meta-language to the object-language 'below' it.

Legitimate name formation and Gödel numbering

Gödel numbering is **not** quotation in the sense indicated above. In fact, Gödel numbering is a one-one **contraction mapping** of an infinite Boolean lattice; the contraction being a mapping of the denumerable part of that lattice onto its skeleton, which is also denumerable.

A contraction mapping of such a kind cannot be defined upon either (a) a finite Boolean lattice, or (b) a complete Boolean lattice, whether finite or infinite. In the finite case, the cardinality of the lattice is strictly greater than the cardinality of the skeleton and so the mapping cannot be one-one. In the second case, the contraction of the denumerable part of the Boolean lattice onto the skeleton induces a contraction mapping on the whole of the Boolean lattice, since from the new skeleton an exact replica of the original lattice may be derived.

Via the completeness theorem for first-order logic any non-denumerable lattice must have a denumerable model (downward Löwenheim-Skolem theorem). Hence, even if a complete lattice is non-denumerable, it has a denumerable model, so a contraction mapping would map all of the lattice onto the skeleton and leave no lattice behind. So the model of a Gödel contraction mapping must be at least of cardinality $\aleph_1$ and hence must be essentially incomplete.

Hence, a Gödel contraction mapping implies that there exists in the lattice as a whole an **inexhaustible** part that acts as a source of lattice points; so that the contraction mapping merely replicates the original structure and acts as a renaming of parts of it. So any Boolean lattice on which Gödel numbering is defined must be incomplete. That is the fundamental reason for the Gödel incompleteness theorem; a logic is sufficiently strong if its model must contain within it an infinite 'boundary' that serves as the inexhaustible source of lattice points. One model of such a lattice is the continuum.

Melampus

References

Birkhoff, Garrett [1940]
Lattice Theory, American Mathematical Society, Rhode Island.
Crawley, Peter and Dilworth, Robert P. [1973]
Algebraic Theory of Lattices. Prentice-Hall INc. New Jersey.
Davey, B. A and Priestley, H.A [1990]
Introduction to Lattices and Order. Cambridge University Press. Cambridge.
by Max Kölbel.)
Gentzen, Gerhard [1969]
The Collected Papers of Gerhard Gentzen. Edited by M.E. Szabo. North-Holland. Amsterdam.
Hume, David [1779]
Dialogues Concerning Natural Religion. Ed. Martin Bell. Penguin. Orig. date of publication: 1779.
Mendelson Elliott [1970]
Schaum's Outline of Theory and Problems of Boolean Algebras and Switching Circuits. McGraw-Hill. New York.
Mendelson Elliott [1979]
Introduction to Mathematical Logic. D. van Nostrand Company. New York.
Ungar, Anthony, M [1945]
Normalization, Cut Elimination, and the Theory of Proofs.