MELAMPUS

Beyond the Axiom of Completeness

Supplement to On the Continuum exploring alternative structures to the continuum other than that provided by the Axiom of Completeness.

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Beyond the Axiom of Completeness

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1. The countable chain condition and the continuum

1.1 The countable chain condition

1.1.1 Definition, countable chain condition

Let (X,T) be a topological space. Then X has the countable chain condition (c.c.c.) iff there is no uncountable family of pairwise disjoint non-empty open subsets of X. (Kunen [1980])

It is a historical accident that this is called the countable <u>chain</u> property; it would be more accurate to call it the countable <u>antichain</u> property. This property is relative to the topology; as Kunen [1980] tells us, "If X is countable, X is trivially c.c.c. If X is uncountable then X may fail to be c.c.c.; for example, if X has the discrete topology, then the singletons form an uncountable disjoint family of open sets." For example, the continuum under the discrete topology has c non-empty open singleton sets. Likewise, the Cantor set with the discrete topology fails the c.c.c. The Cantor set is generated from a base partition of ω elements by cardinal exponentiation; $\mathbf{2}^{\omega} = \mathbf{P}(\omega)$ so has another topology which is c.c.c. Any set with a topological base $|X| > \omega$ is automatically not c.c.c.

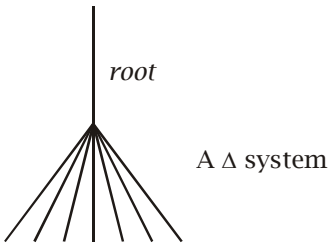
1.1.2 Definition, almost disjoint

Let κ be an infinite cardinal. Let $x, y \subset \kappa$. Then x,y are said to be *almost disjoint* iff $|x \cap y| < \kappa$. An *almost disjoint family* is an $A \subset P(\kappa)$ such that $(\forall x \in A)(|x| = \kappa)$ and any two distinct elements of A are almost disjoint. A *maximal* almost disjoint family is an almost disjoint family with no almost disjoint family B properly containing it. (Kunen [1980])

Let a collection of almost disjoint sets form a poset. Then there are chains in this poset representing sub-posets or forcing conditions which overlap on some common set of (finite) conditions; they correspond to different paths in a common ideal. The overlap is the node that defines the ideal. The node defines a "root" of the system.

1.1.3 Definition, delta system

A family A of sets is said to be a Δ**-system** or a **quasi-disjoint** family iff there is a fixed set r, called the *root* of the Δ-system, such that $a \cap b = r$ whenever a and b are distinct members of A. It is possible that the root is null. (Kunen [1980])



Because $a \cap b = r$ the branches are compatible. A countable collection of delta systems does not violate the countable chain condition.

1.1.4 Definition, second countable

A second countable space has a countable open base. (Simmons [1983])

1.1.5 Theorem

Every separable metric space is second countable. (Simmons [1983])

1.1.6 Lemma

If X is separable, then X has c.c.c. (Kunen [1980])

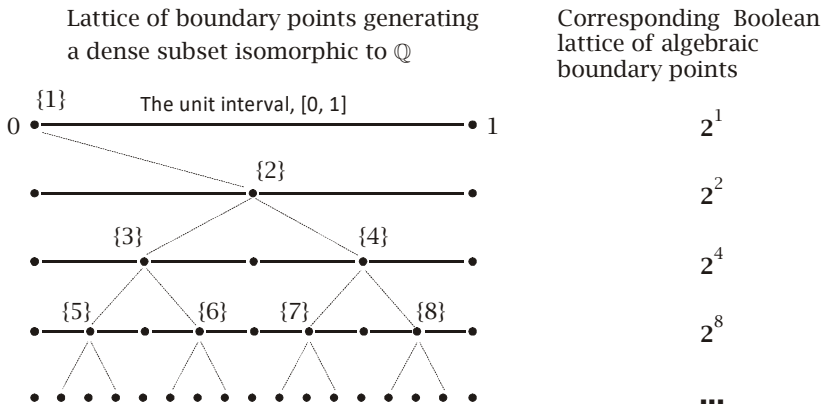
<u>Proof</u>

Let $D \subset X$ be dense and countable. Suppose $\{U_\alpha : \alpha < \omega_1\}$ is a collection in X of open, non-empty, disjoint sets. Then choose $d_\alpha \in U_\alpha \cap D$. Then the d_α would constitute a set of size ω_1 of distinct elements, contradicting the fact that D is countable.

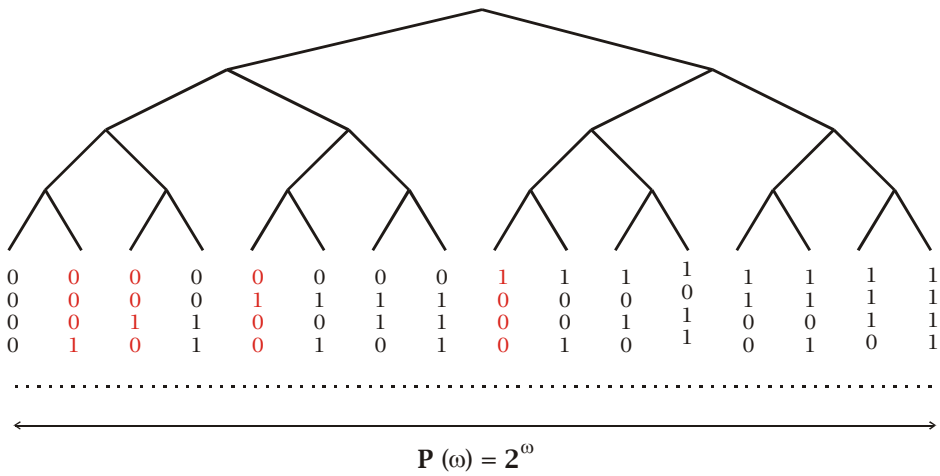
<u>Corollary</u>

$\mathbb{R}$ must be c.c.c. as it is separable.

Consider the lattice of boundary points generating a dense subset of the unit interval.



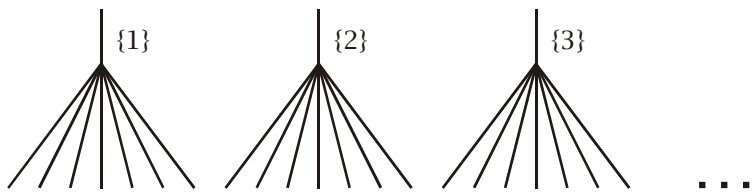
This diagram illustrates the fact that $\mathbf{Fin} \cong \mathbf{2}^{\mathbb{Q}} \cong \mathbf{2}^{\mathbb{N}} \sim \mathbf{2}^{<\omega}$ is non-atomic. The downward process of adding new divisions of the unit interval is not completed from within the tree. If we take the limit and allow the branches to reach length ω then we generate not ω atoms, but 2^{ω} atoms. This illustrates that there are 2^{ω} ultrafilters in **Fin** and from within **Fin** the atoms are perceived to be points of the real line. However, if a space is separable then it has a countable basis of open sets (being second countable). This is illustrated by the tree for the corresponding Boolean lattice.



Here the separable set appears as a set of linearly independent vectors – atoms – a subset of size ω within the entire tree $\mathbf{2}^{\omega}$. This diagram does not display transparently the relation between the atoms (vectors) and the other elements that depend on them. So we may redraw the tree as a collection of delta systems. Each delta system will have as root an atom. Let

$(1,0,0,0,\ldots) \leftrightarrow \{1\}$
$(0,1,0,0,\ldots) \leftrightarrow \{2\}$

then we have delta systems: -



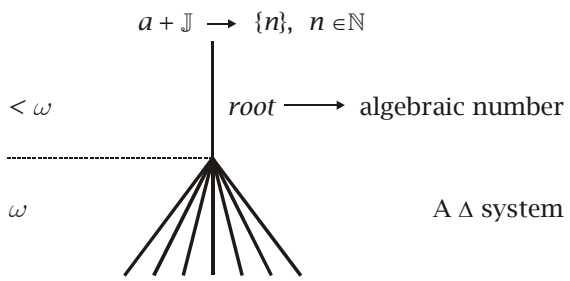
Any separable space may be partitioned into a countable collection of delta systems. The countable collection comprises its skeleton.

We need a further word about this partition. At first glance it does not appear that the separate delta systems are truly disjoint. Consider, for example, the situation in a finite Boolean lattice (algebra), $\bigcap B \neq \varnothing$, where the branches of separate atoms may be linearly combined:

$$\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

(Here the '+' indicates the operation of addition in the Boolean ring to which the Boolean algebra is isomorphic.) This shows that the branch corresponding to the vector $(0,0,1,1)^t$ belongs to both delta systems with roots (atoms), $(0,0,0,1)^t$ and $(0,0,1,0)^t$; delta systems overlap. However, in the infinite lattice, $\mathbf{2}^\omega$, this situation does to obtain. Because the branches are actually infinite an actually infinite sequence of 0s precedes the first 1 in any vector representing an atom: $\left(0,0,\overbrace{\ldots\ldots\ldots\ldots}^{\omega},0,1,0,0,\ldots\right)^t$, so the other branches depend uniquely on a given atomic basis vector. Another way of demonstrating this is to observe that the countable skeleton of the real line is a representing set for $\mathbb{Q}$, the dense set of rationals in $\mathbb{R}$. So our collection of disjoint delta systems is isomorphic to the quotient algebra, $\frac{[0,1)}{\mathbb{J}}$, where $\mathbb{J}$ is the set of irrationals. The cosets (partitions) of this quotient algebra are $a+\mathbb{J}=\{a+\xi : a \text{ is algebraic and } \xi\in\mathbb{J}\}$. Each coset is disjoint form the other and contains continuum many members, each corresponding to a branch of the delta system of which $a+\mathbb{J}$ is the root.

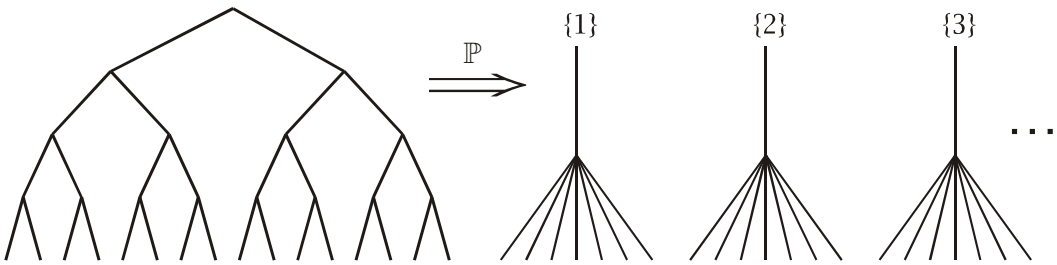
Any algebraic number may be the root of a delta system; the set of all algebraic numbers is countable. So the roots of the delta systems may be regarded as of any countable ordinal length, $<\omega$. In the above diagram the labels $\{1\},\{2\},\{3\},\ldots$ are names of any dense partition of the real line, so may denote chains of any ordinal length $<\omega$.



Cohen reals sharing the same root

As the diagrams indicate, this is the skeleton of the boundary points in **Fin**. The cofinite sets of **Cofin** has its own collection of delta systems, the roots of its skeleton are co-atoms.

What the above argument illustrates is that the existence of a partial order $\mathbb{P}$ satisying the c.c.c. amounts to a mapping or transformation of the Cantor set, $\mathbf{2}^{\omega}$, into a countable family of delta systems.



There arises the question: where does the partial order $\mathbb{P}$ exist? In this case, $\mathbb{P}$ is generated by the potentially infinite part of the Cantor set, $\mathbf{2}^{<\omega}$, as branches encoding an algebraic number. The algebraic numbers are generated as potentially infinite sequences, but in set theory (ZFC) we also assume that they form a complete, actually infinite totality, and when we factor through by this set, we get a transformation of the Cantor set as illustrated above.

There thus arises the possibility of other transformations of the Cantor set that display some structure not otherwise manifested in the mere tree representation; we may impose partial orders on the Cantor set by means of axioms that require the Cantor set to display a whole range of these different structures. There will be constraints on this process, so that not all partial orders may coexist in a given model.

It is a moot question: must such partial orders already pre-exist in the potentially infinite part of the Cantor tree? This emerges as a kind of coherence issue. By adopting some kind of forcing axiom we simultaneously embed the partial order into the potentially infinite tree and generate the map that transforms it and displays the substructure of the complete Cantor tree. We may think of this process as one of adding something from outside or of displaying what was already internal to the structure, but not previously known. The two conceptions are equivalent, because really it is a question of choosing one determination of the Cantor set as opposed to another, and the Cantor set is under-determined within ZFC.

In order to move in the direction of the discussion of these transformation, we some more technical gadgetry.

1.1.7 Delta-system lemma.

If A is any uncountable family of finite sets, there is an uncountable $B \subset A$ which forms a Δ-system. (Kunen [1980])

<u>Proof</u>

The proof given by Kunen is by means of a more general theorem and it is a murky nightmare. However, he says there is a direct and easy proof of the theorem and gives a hint in exercise 1.

<u>The hint in exercise 1</u>

For some *n*, we are allowed to assume $(\forall x \in A)(|x| = n)$. The proof proceeds by induction on *n*. There is an uncountable $B \subset A$ such that either $\bigcap B \neq \varnothing$ or *B* is pairwise disjoint.

<u>Solution</u>

Suppose $n = 1$. Then A is an uncountable collection of singletons. These must be pairwise disjoint. These form a Δ-system in which the root is null. We have $B = A$. Now suppose that for all $n \leq k$ the property holds. Let $(\forall x \in A)(|x| = k+1)$. Then for each *x*, $x = y \cup \{t\}$ where $|y| = k$. Then by the induction hypothesis, $C = \{y : |y| = k\}$ is a delta system with uncountable $B \subset C$ such that either $\bigcap B \neq \varnothing$ or *B* is pairwise disjoint. If the latter, then the $x = y \cup \{t\}$ form either an uncountable collection of pairwise disjoint sets or a collection *B* such that $\bigcap B \neq \varnothing$. If $\bigcap B \neq \varnothing$, then we have $B' \subset A$ $\bigcap B' \neq \varnothing$.

Kunen tells us that "the question of whether the product of any two c.c.c. spaces must be c.c.c. is independent of ZFC". It is true under $MA + \neg CH$ but it fails if CH holds or if Suslin's Hypothesis fails. However, there is the following result.

1.1.8 Theorem

Let $X_i (i \in I)$ be spaces such that for every finite $r \subset I$ $\prod_{i \in r} X_i$ is c.c.c. Then $\prod_{i \in I} X_i$ is c.c.c.

On the face of it this is a remarkable and even counter-intuitive theorem. The resolution to this problem is contained in the above remarks. It depends on understanding that the definition of c.c.c. enables one to define on the infinite product a topology such that it is also c.c.c. In fact, the wording of the theorem as stated in Kunen is not exact, it should read, "there exists a topology *T* on $\prod_{i \in I} X_i$ that is c.c.c."

<u>Proof</u>

Let $\{U_\alpha\}$, $\omega < \alpha < \omega_1$ be pairwise disjoint non-empty sets in $\prod_{i \in I} X_i$. We assume that the collection $\{U_\alpha\}$ is uncountably infinite because we are aiming to show that this results in a contradiction. We may also assume that the $\{U_\alpha\}$ form a basis for $\prod_{i \in I} X_i$. (By showing that the assumption that the basis is uncountable leads to contradiction the proof actually shows that any basis in this case must be countably infinite.) Each U_α is defined by a countable set of coordinates $a_\alpha \subset I$.

$\prod_{i \in I} X_i$ is the infinite product of ccc spaces. Each space provides a coordinate. So why not an infinite collection of coordinates? This is because each U_α has some coordinates in common with each other. The rule about the coordinates is given by: $U_\alpha \cap U_\beta = \varnothing$ iff $a_\alpha \cap a_\beta \neq \varnothing$. The overlap of all of the $\{U_\alpha\}$ constitute the root of the system. Formally, the existence of the root is given by the Δ-system lemma. Let the root be *r*. This is anticipated in the formulation of the statement of the theorem with $\prod_{i \in r} X_i$ being c.c.c. Let

$$\pi : \begin{cases} \prod_{i \in I} X_i \to \prod_{i \in r} X_i \\ U_\alpha \to \pi(U_\alpha) \end{cases}$$

be the projection of the basis onto the root of the system. It may be shown that this is a bijection. Since $\{U_\alpha\}$ is uncountable then the $\{\pi(U_\alpha)\}$ are uncountable. This contradicts the c.c.c. condition for the root. Hence the theorem is proven.

1.1.9 Definition

Caliber ω_1. Let $\{U_\alpha\}, \alpha < \omega_1$ be a collection of open subsets of *X*. Then there is an uncountable $A \subset \omega_1$ such that $\bigcap_{\alpha \in A} U_\alpha \neq \varnothing$.

No caliber ω_1 topology can have a basis with more than ω members. So what we have shown is that in the above theorem $\prod_{i \in I} X_i$ has such a caliber. If $\bigcap_{\alpha \in A} U_\alpha \neq \varnothing$ then the $\{U_\alpha\}, \alpha < \omega_1$ are not disjoint; hence, not a basis. We have the following implications: -

separable $\Rightarrow$ caliber $\omega_1 \Rightarrow$ ccc

1.1.10 Result

1. $\mathbf{2}^\kappa$ has a c.c.c. topology for any κ where $\mathbf{2} = \{0,1\}$ with the discrete topology. 2. When $\kappa > \mathbf{2}^\omega$ this is an example of a c.c.c. space that is not separable.
2. $\mathbf{2}^\kappa = \{0,1\}^\kappa$, $\kappa > 2^\omega$ is c.c.c. but not separable.

The first part of this is an immediate consequence of the preceding theorem. Kunen does not offer a proof of the second part, but a hint.

<u>Hint</u>

If $D \subset \mathbf{2}^\kappa$ is countable, show that there are $\alpha < \beta$ such that $(\forall f \in D)(f(\alpha) = f(\beta))$.

<u>Solution</u>

This is basically an application of the pigeon-hole principle. Firstly, looking at the case of the Cantor set with what I'll call the ω-topology, and assuming the CH, then we have $2^{\aleph_0} = \aleph_1$ so taking two distinct sets (ordinals) $\omega < \alpha < \beta < 2^{\omega}$ then α, β must both be countably infinite, of cardinality $\aleph_0$, and hence there must be a map f onto the ω-topology such that $U_\alpha = f(\alpha) \neq f(\beta) = U_\beta$, so they are always separable.

Now let's apply this to the given case. We have $\kappa > 2^{\omega}$ so this entails that there must be at least three distinct cardinalities involved. The cardinality of $2^{2^{\omega}}$ is undecided; it is possible that $2^{\aleph_1} = \aleph_1$, but $\kappa > 2^{\omega}$ forces $2^{\kappa} \geq \aleph_2$. So we must have sets $\alpha, \beta \in 2^{\kappa}$ of cardinality $\aleph_1$. If $X = 2^{\kappa}$ has the c.c.c. topology then the topology in which this is the case can have only $\aleph_0$ disjoint non-empty open subsets. So no matter how we map X onto this topology, there must be open sets α, β of cardinality $\aleph_1$ such that $U_\alpha = f(\alpha) = f(\beta) = U_\beta$, and these points are not separable. Furthermore, while X may have a topology which is not c.c.c., it will always have one that is c.c.c., which is the topology defined by a base of ω segments.

1.2 Collapsing cardinals

Here I explore how to construct within ZFC alternative models of the continuum. The first stage of this is to revisit Suslin's question: -

1.2.1 Suslin's question [1920]

Can the real line be characterised as the unique complete, dense in itself linear order with the countable chain condition?

Strepans [X] comments, "Suslin was aware that the real line could be characterized as the unique separable, complete, linear order with neither end points nor isolated points. So Suslin is asking whether the separability condition can be replaced by the seemingly weaker one that all disjoint families of open sets are countable."

Following Kunen [1980] we let M be a countable transitive model of as much of ZFC as is required. (That is, usually, all of ZFC less the full power set axiom.) We consider a forcing with a poset of finite partial functions from one set I into another set J.

1.2.2 Definition

$\mathrm{Fn}(I,J) = \{p : |p| < \omega \text{ and } p \text{ is a function and } \mathrm{dom}(p) \subset I \text{ and } \mathrm{ran}(p) \subset J\}$

Let $\mathrm{Fn}(I,J)$ by ordered by $p \leq q \leftrightarrow q \subset p$.

Thus $\mathrm{Fn}(I,J)$ is a poset with largest element **1** being the empty function. (Kunen [1980])

1.2.3 Lemma

Conditions:

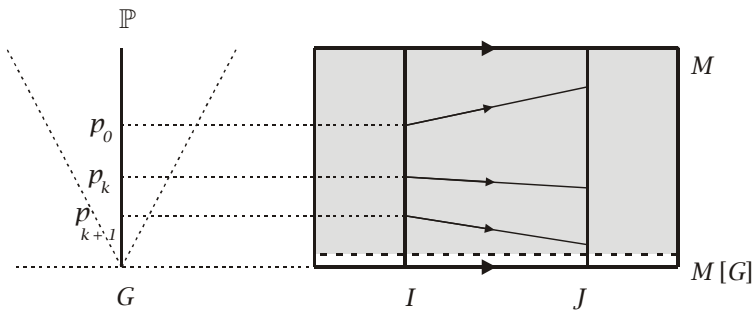
1. M is a model of ZFC; $I, J \in M$ are subsets of M.
2. There exists a partial order $\mathbb{P}$ such that $(\forall p \in \mathbb{P})(p \in \mathrm{Fn}(I,J))$. That is, p is a function from I into J. This function is not necessarily total, and not necessarily onto.
3. There exists a filter G in $\mathrm{Fn}(I,J)$ such that $\bigcup G$ is a function with $\mathrm{dom}(\bigcup G) \subset I$, $\mathrm{ran}(\bigcup G) \subset J$
4. $D_i = \{p \in \mathrm{Fn}(I,J) : i \in \mathrm{dom}(p)\}$ is dense for all $i \in I$.
5. G is generic; i.e. $G \cap D_i \neq 0$ for all $i \in I$.

If all these conditions are fulfilled, then

$\bigcup G$ is a function from I <u>onto</u> J.

This is a general tool as no actual forcing argument has been provided. It is conditional. If a given forcing argument satisfies these conditions, then the conclusion also follows. There appears to be no actual proof of this lemma in Kunen.

Nonetheless, the following diagram illustrates this process. In it, if the conditions are met, we have: If $\bigcup_{n\in\omega} I_n = I$ then $\bigcup_{n\in\omega} J_n = J$, which is the real big statement.



If the partial order is dense, and hence below any point there lies a (potentially) infinite chain, then if we exhaust the chain (by taking the actual infinite) then we must exhaust the range as well. In the above diagram I have visualised the partial order as defining a monotonic increasing sequence of functions, which given AC is acceptable.

Initially, J may be a set seen from M has having larger cardinality. The addition of the partial order – equivalent to a real number generator – maps some of I onto all of J, and so establishes a one-one correspondence between them. In the absence of the real number

generator, J cannot be compared with I except to say that it is larger, so it has larger cardinality. The addition of the generator gives them the same cardinality in the larger model.

1.2.4 Cardinality is not absolute

Thus cardinality is not absolute, and that what appears larger in one model (or universe) is the same size in another.

The addition of any specific real number generator is a countable "one-off" event. This can be repeated, and by this process a denumerable set of generic – i.e. transcendental – real numbers can be added – that is, named, from "within" *M*. However, by Cantor's theorem (anti-diagonalisation) the set of all real number generators is not denumerable so we still have $c > \aleph_0$.

1.2.5 Process for collapsing cardinals

The notion of a cardinal is not absolute for *M*, $M[G]$. Let κ be an uncountable cardinal of *M*. Let $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$ and let G be $\mathbb{P}$-generic over *M*. Then $\bigcup G \in M[G]$ by absoluteness of union, and G is a function from ω onto κ. Hence κ is a countable ordinal in $M[G]$.

We start with a sequence of functions in a poset, $p_0, \ldots, p_k, \ldots$ that maps part of ω into an ordinal $\kappa > \omega$. We append to M a generic function $\bigcup G$ which the lemma shows maps all of ω onto κ; so we have a bijection in $M[G]$ between these two ordinals; the larger ordinal has been collapsed. In $M[G]$ the order of I is now $\omega + 1$ but since we have $\omega + 1 \sim \omega$ we have not increased the cardinality of *I in this way.*

Again, the obvious point is that this is all conditional. We have not demonstrated any particular method of forcing that adds any functions, simply that it is consistent with ZFC to do so. This illustrates the fecundity of ZFC set theory, or alternatively, its weakness. It spawns many models. It is not categorical.

$\mathbb{P}$ is said to *collapse* κ. However, we will demonstrate shortly that a poset that is ccc preserves cofinalities and cardinalities. In this example the partial order $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$ is not ccc, since it collapses cardinals.

This forcing is not of the type $\mathbb{P} = \mathrm{Fn}(I, J)$ where J is countable. So the lemma that proves that a forcing is c.c.c. does not apply in this case. To collapse cardinals in general one uses forcings that are not c.c.c. The effect of this partial order is to transform $\mathbf{2}^{\omega}$ into $\mathbf{2}^{\kappa}$ and thus demonstrate that $\mathbf{2}^{\omega} = \mathbf{2}^{\kappa}$ in the generic extension; in the ground model $\kappa > \omega$ but in the extension $\kappa = \omega$.

1.3 Alternative models of the continuum

Because the structure $2^{\omega} = \{0,1\}^{\omega}$ is not categorical, it may have different structures with different topologies – each depends on what combinatorial principles we adopt. In the first instance we consider those models of the arithmetical continuum that continue to be c.c.c. Thus, to consider Suslin's question – so far as the Axiom of Completeness is concerned there is only one categorical model of the continuum, and that must be separable. However, as the arithmetical continuum is not an object given directly to intuition (it not being a claim here that there are any such objects), the Axiom of Completeness becomes an empirical hypothesis. Ultimately, if it is valid, it will be because observation supports it. Therefore, it is admissible to consider alternative models of the continuum. An obvious starting point is to consider a model of the continuum that retains the c.c.c. condition, while possibly dropping the requirement that it is separable. The c.c.c. condition is equivalent to the requirement that the model has a countably infinite skeleton.

This is a very interesting point of departure. The development of set theory is initiated by a desire to investigate the arithmetical continuum, which it is supposed is already an unambiguous structure that ought to have a categorical model in ZFC. However, it emerges that ZFC enables one to construct a whole range of alternative structures. The c.c.c. condition is a pragmatic one as much as anything. Without it, the structure is in danger of becoming ill-defined. It is already mishmash on account of the so-called existence of singular cardinals, which really upset the apple cart and make results very difficult to develop. Removal of the c.c.c. notion from the forcings would increase this tendency to ill-defined structure.

Another reason for focusing on forcings that obey the c.c.c. condition, is that those that do not are rather dull; collapsing cardinals makes no point in a theory that invents cardinals precisely to use the distinctions between them to introduce fine structure into a given model.

To each model of the continuum there corresponds a collection of forcings, each of which in turn depends on what posets we allow into the set theoretical universe as a whole. We cannot allow them all, for otherwise we shall have an inconsistent theory. The following lemma provides a general condition on posets specifying when they have the c.c.c.

1.3.1. Lemma

If I is arbitrary and J is countable, then $\mathbb{P} = \mathrm{Fn}(I,J)$ has c.c.c.

<u>Proof</u>

Since I is arbitary, by passing to a subset we may assume that $|I| = \omega_1$.

Let $p_\alpha \in \mathbb{P} = \mathrm{Fn}(I,J)$, $\alpha \in \omega_1$.

Let $A_\alpha = \mathrm{dom}(p_\alpha)$ and $P_\alpha = \mathrm{ran}(p_\alpha)$.

By the Δ-system lemma there is an uncountable $X \subset \omega_1$ such that $\{A_\alpha : \alpha \in X\}$ is a Δ-system with root r.

So $P_\alpha \restriction r$ is countable, because it is the inverse image of a countable set.

Then there is an uncountable $Y \subset X$ such that $P_\alpha \restriction r$, $\alpha \in Y$ are all the same.

So the P_α for all $\alpha \in Y$ are all compatible.

Thus, there can never be a family $\{P_\alpha : \alpha < \omega_1\}$ of incompatible conditions.

Kunen remarks: "The importance of c.c.c. in forcing is the following lemma, which gives us a way of approximating, within *M*, any function which appears in $M[G]$."

1.3.2 Approximation Lemma

Let $\mathbb{P} \in M$ be c.c.c. in *M*. Let $A, B \in M$. Let G be $\mathbb{P}$-generic over *M*. Let $f : A \to B, f \in M[G]$.

Then

(1) There is a map: $F \to \mathbf{P}(B)$ such that $F \in M$

$(\forall a \in A)(f(a) \in F(a))$.

(2) $(\forall a \in A)(|F(a)| \le \omega)$ in *M*.

<u>Proof</u>

The proof is in two parts.

(1) Proof of the existence of an approximating function *F* to *f*. This arises from the fact that *f* is forced by conditions in *M* so really only exists because there is an *F* in *M* that approximates it. This would apply regardless of whether the poset giving the forcing is ccc or not. The technical details depend on the gadgetry of naming elements of the extension $M[G]$ from within *M*, and as I have described these, I omit them here as well. However, the result is intuitively obvious and there is almost nothing to prove.

(2) It is in this part that the c.c.c. condition is used. This is an outline of the proof. Let b, b' be two distinct elements of *F*. That is: -

$b, b' \in F(a), b \ne b'$.

Then they force inconsistent statements, and hence there is a function $Q \in M$ such that $Q(b) \perp Q(b')$ - that is, they are incompatible statements. Then $\{Q(b) : b \in F(a)\}$ is an antichain in $\mathbb{P}$. Since $Q \in M$ and $\mathbb{P}$ is c.c.c. in *M*, we have $|F(a) \le \omega|$ in *M*.

1.3.3 Definition, preserving cardinals

$\mathbb{P}$ preserves cardinals if whenever *G* is $\mathbb{P}$ generic over *M*, we have for all β: -

β is a cardinal in $M \Leftrightarrow$ β is a cardinal in $M[G]$.

1.3.4 Theorem

If $\mathbb{P} \in M$ is c.c.c., then $\mathbb{P}$ preserves cardinalities.

<u>Proof</u>

ω is absolute so the preservation of cardinals is only problematic for $\beta > \omega$.

The reverse implication is automatic; i.e.

β is a cardinal in $M[G] \Rightarrow$ β is a cardinal in *M*.

This is because any function in $M[G]$ from a smaller ordinal onto β would be in *M*.

For the forward implication, we have to show that if $\beta > \omega$ is a cardinal in *M* then it is a cardinal in $M[G]$. The proof is by contradiction. Suppose β is not a cardinal in $M[G]$. Then there exists a function $f \in M[G]$ such that $f : \beta \to \gamma$ where γ is an ordinal $< \beta$. Since $\beta \in M$, then $\gamma \in M$. Then by the approximation lemma, there is an $F \in M$ such that $\gamma = f(\beta) = F(\beta)$ and $|\gamma| = |F(\beta)| \le \omega$. So F maps $\beta \to \gamma$ in *M* as well. Hence β is not a cardinal in *M*.

1.3.5 Theorem

If $\mathbb{P} \in M$ is ccc, then $\mathbb{P}$ preserves cofinalities and hence cardinalities.

1.3.6 Definition, preserving cofinalities

If $\mathbb{P} \in M$, $\mathbb{P}$ **preserves cofinalities** iff whenever *G* is $\mathbb{P}$-generic over *M* and γ is a limit ordinal in *M*, $\mathrm{cf}(\gamma)^M = \mathrm{cf}(\gamma)^{M[G]}$. (Kunen [1980])

Preservation of cofinalities is a stronger property than preservation of cardinals, and implies it.

1.3.6 Lemma

Let $\mathbb{P} \in M$. Suppose that whenever *G* is $\mathbb{P}$-generic over *M*, and κ is a regular uncountable cardinal of *M*, then κ is regular in $M[G]$. Then, $\mathbb{P}$ preserves cofinalities.

For the proof see Kunen [1980].

1.3.7 Theorem

If $\mathbb{P} \in M$ and $\mathbb{P}$ has the c.c.c. in *M*, then $\mathbb{P}$ preserves cofinalities and hence cardinalities.

<u>Proof</u>

Suppose not. Then by Lemma 1.3.6 there exists a $\kappa \in M, \kappa > \omega, \kappa$ regular in M and κ not regular in $M[G]$. Therefore, there is an $\alpha < \kappa$ and $f \in M[G]$, f maps α cofinally into κ.

Lemma 5.5 uses the ccc; from this lemma if follows that there is an $F \in M$ such that

$F : \alpha \to \mathbf{P}(\kappa)$

$(\forall \xi < \alpha)(f(\xi) \in F(\xi))$

$(\forall \xi < \alpha)(|F(\xi) \le \omega|)$ in M

Let $S = \bigcup_{\xi < \alpha} F(\xi)$. Then $S \in M$ and S is unbounded in κ.

Applying in M the fact that the union of $|\alpha|$ countable sets has cardinality $|\alpha|$.

$|S| = |\alpha| < \kappa$ in M.

Therefore, κ is not regular in M.

1.3.8 Theorem

There exists a model in which CH is false.

<u>Proof outline</u>

Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$.

Let G be $\mathbb{P}$-generic over M.

Then $\bigcup \mathbb{G} : \kappa \times \omega \to 2$.

Therefore, $M[G]$ contains a κ sequence of distinct functions from $\kappa \times \omega$ into $\mathbf{2} = \{0,1\}$.

By absoluteness this sequence is in $M[G]$.

It may be shown that each function in this sequence is distinct. Hence

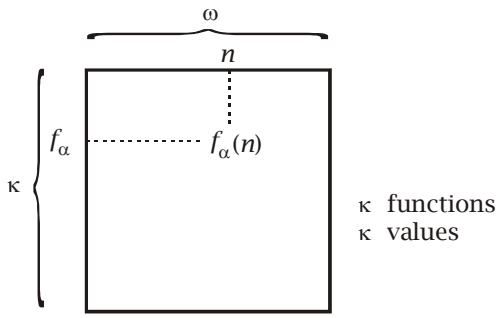
If $\kappa \in M$ and G is $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over M then $2^{\omega} \ge |\kappa|$ in $M[G]$.

Hence, with $\kappa = \omega_2$ in M we have $2^{\omega} \ge \omega_2$ in $M[G]$, whence CH fails in $M[G]$.

As Kunen remarks, "We may think of G as coding a κ-sequence of functions from ω into 2." (p.205). Thus $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ defines a characteristic function: -

$$f_{\alpha}(n) = \begin{cases} 1 & \text{if } f_{\alpha}(n) \text{ is defined and } f_{\alpha}(n) \in G \\ 0 & \text{if } f_{\alpha}(n) \text{ is undefined or } f_{\alpha}(n) \notin G \end{cases}$$

Then G is a coding for a real number, $x \subset \omega$ that maps $\kappa \times \omega$ to $\mathbf{2}$. One would "prefer" to see $\mathrm{Fn}(\kappa, 2^{\omega})$, that is, $f_{\alpha} : \kappa \to \mathbf{2}^{\omega}$ to give it as a mapping from κ into 2^{ω} but this is not possible because in M the collection of functions is not necessary total, so in M, as above, it is possible that $\kappa \ge 2^{\omega}$.



The mapping $\bigcup G : \kappa \times \omega \to 2$ defines a sequence $f_\alpha(n) : \kappa \to 2^\omega$. There are ω assignments for each of the κ functions and $\mathrm{Im}(f) \subseteq 2^\omega$. In this case the generic object is a mapping <u>into</u>, and we have $|\kappa| \le 2^\omega$ in $M[G]$. In M the mapping of $\kappa \sim \kappa \times \omega$ into 2^ω is not necessarily total, so it is possible in M that $\kappa \ge 2^\omega$ - or anything – the size of κ relative to 2^ω is not fixed. Once we add the generic function, as per lemma X: -

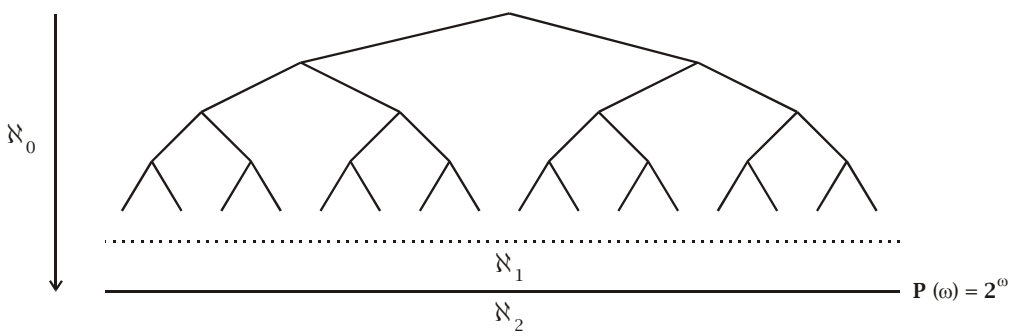
If $\bigcup_{n\in\omega} I_n = I$ then $\bigcup_{n\in\omega} J_n = J$

Here as: -

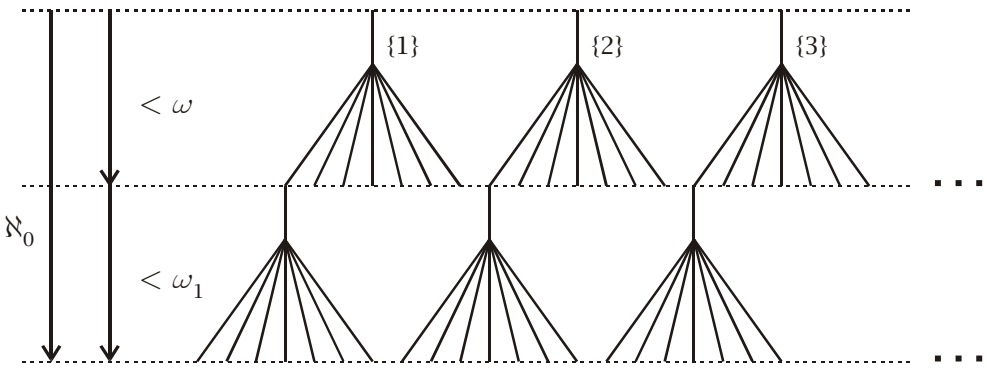
If $\bigcup_{\alpha\in\kappa} \mathrm{dom}(f_\alpha) = \kappa$ then $\bigcup_{n\in\omega} \mathrm{Im}(f_\alpha) = \mathbf{2}^\omega$

$\bigcup G : \kappa \times \omega \to 2$ becomes a bijection of κ onto 2^ω, so in this way a model that falsifies CH can be constructed. The other point of the $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ formulation is so that one can apply directly to it the lemma coming up concerning the cc.c. condition, and hence prove that the function preserves cardinalities.

Suppose we have $\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega, 2)$, which entails $2^\omega \ge \omega_2$. To be definite, let $2^\omega = \omega_2$, so we have three cardinalities: $\aleph_0 < \aleph_1 < \aleph_2 = 2^{\aleph_0}$. This is also a model of the continuum which is c.c.c. so has a skeleton of ω atoms. Then the Cantor tree has the distinct families of subsets of all three cardinalities: $\aleph_0 < \aleph_1 < \aleph_2 = 2^{\aleph_0}$. We may show this diagrammatically as follows:



Before the Cantor tree has reached down to the level of actually ω iterations, it has generated a distinct set of width $\aleph_1 < \aleph_2$. At the ω th iteration, the width is $\aleph_2 = 2^{\aleph_0}$. As the tree is still c.c.c. the skeleton has ω atoms. So the tree structure may be transformed into a structure that is visualised as: -



This tree structure requires Martin's Axiom in the form $\mathrm{MA}(\omega_1)$. As $\aleph_2 = 2^{\aleph_0}$, we have $\neg\,\mathrm{MA}(\omega_2)$.

1.3.9 Properties of this model

Defined by the existence in the ground model of the partial order $\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega, 2)$.

1. The product of any two c.c.c. spaces is c.c.c. (Theorem 2.24 Kunen)
2. It is consistent with the Axiom of Projective Determinacy, which also implies $\aleph_2 = 2^{\aleph_0}$.
3. $\mathrm{MA}(\omega_1) \Rightarrow 2^{\omega_1} = 2^{\omega_0}$. (Theorem 2.18 Kunen)
4. $\bigcup_{\alpha<\omega_1} M_\alpha$ where M_α are first category are first category. It also has Lebesgue measure 0. (Theorems 2.20 and 2.12 Kunen, p. 58 and 59)
5. Suslin's hypothesis (SH) holds. That is, there is no Suslin line.
   Therefore, in this model the Suslin line generated by the extension points, the S transcendentals must be destroyed. One can see intuitively that the open not-null sets that became the amoeba S transcendentals after actually ω iterations have been populated with further reals which are defined by ultrafilters subject to $\mathrm{MA}(\omega_1)$. So the Suslin tree has been destroyed. There is no Suslin tree. There are no ω_1-Suslin trees. (Kunen, Lemma 5.14)
6. In this model the Mahler classification of transcendental numbers is not valid. For example, any Liouville number in the Maher classification is the root of a family of real numbers and not a real number itself. It might be called "quasi"

algebraic. Essentially, in this model we are extending the notion of an algebraic number beyond polynomials of degree ω.

1.4 Demystification of the structure of the continuum.

One of the problems of the study of the continuum is that the results come out as inequalities: for example, our poset $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ yields, $\aleph_2 \le 2^{\aleph_0}$ and not the strict $\aleph_2 = 2^{\aleph_0}$. This is because the completed (actual) infinities are constructed from below as the limits of potential infinities, so the notion of cofinality rather than cardinality is fundamental in ZFC. However, we must also ask, if we have $\aleph_2 \le 2^{\aleph_0}$, which is consistent with $\aleph_2 < \aleph_3 < \ldots < 2^{\aleph_0}$, then if additionally there are actually sets of cardinality $\aleph_3$ how are these generated? We see from all the preceding discussion that sets of any cardinality are added to the ground model as generic extensions precisely by posets $\mathbb{P}$ corresponding to forcings. If the posets do not exist in the ground model, then the corresponding set of given cardinality is not added. Another way of looking on this point is by studying the following lemma of Kunen: -

1.4.1 Lemma

Let κ be an infinite cardinal of *m* such that $\left(\kappa^{\omega} = \kappa\right)^{M}$, and let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Let *G* be $\mathbb{P}$ generic over *M*. Then $\left(2^{\omega} = \kappa\right)^{M[G]}$. (Kunen [1980])

Kunen remarks, "In particular, if *M* satisfies GCH, then in *M*, $\kappa^{\omega} = \kappa$ whenever $\mathrm{cf}(\kappa) > \omega$ ..." In this lemma we obtain an exact cardinality for the continuum, though only on the assumption that the generalised continuum hypothesis holds in the ground model, *M*. However, we have seen that subject to the Axiom of Completeness the Continuum Hypothesis does hold in the ground model provided by the Derived Set. Therefore, it seems appropriate to assume that GCH holds in the minimum model and look upon the other models as constructed out of these as generic extensions subject to specific forcings. In that way the inequalities become equalities, and we can talk in terms of a hierarchy of definite models of the continuum. Furthermore, once we realise that the structure of the arithmetical continuum is ultimately an empirical question, then any hint of arbitrariness about this approach will be eliminated by the need to test the validity of the chosen model against its observable consequences. This is a project beyond the scope of this paper.

1.4.2 Summary

In this section we have discussed two main examples: -

1. $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$

Not c.c.c. In the ground model, *M*, we have $\kappa > \omega$. In the generic extension $M[G]$ we have $\kappa = \omega$.

2. $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$

Is c.c.c.

In the ground model, *M*, we have $\kappa > \omega$ and $\kappa, 2^{\omega}$ are incomparable.

In the generic extension $M[G]$ we have $\omega < \kappa = 2^{\omega}$.

Since κ can be anything, this falsifies CH.

The first example is not a likely model of the continuum. It is of theoretic interest because it shows that the notion of cardinality is not absolute for *M*, or absolute in ZFC. The second example is a possible model of the continuum, and has properties that make it pragmatically desirable. However, we sacrifice the simplicity provided by the ground model provided by the Axiom of Completeness in which CH holds. Mahler *S* numbers disappear and are replaced by clusters of other reals.

2. Fine-tuning the structure of the continuum

2.1 Cardinal invariants and the Cichón diagram

The fine structure of the arithmetical continuum is characterised by a collection of cardinal numbers, all of which must be greater than $\aleph_1$ in size.

2.1.1 Definitions

Let J be an ideal. Then: -

$\mathrm{add}(J)$ is the least cardinality of a family $\mathcal{A}$ of subsets of J such that $\bigcup \mathcal{A} \notin J$.

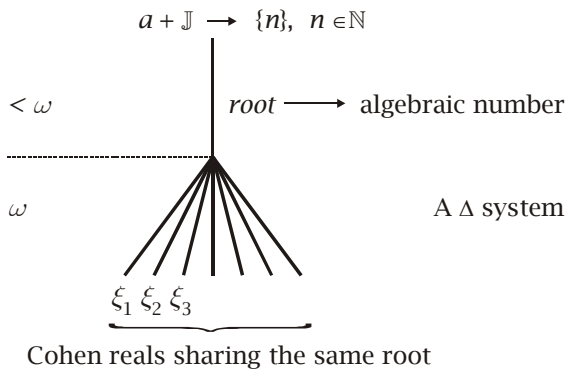
$\mathrm{cov}(J)$ is the least cardinality of a family $\mathcal{A}$ of subsets of J such that $\bigcup \mathcal{A} = \mathbb{R}$.

$\mathrm{non}(J)$ is the least cardinality of $x \subseteq \mathbb{R}$ so that $x \notin J$..

$\mathrm{cof}(J)$ is the least cardinality of a family $\mathcal{A}$ of subsets of J such that:-

$(\forall x \in J)(\exists a \in \mathcal{A})(x \subseteq a)$.

In general we are concerned with ideals $\mathcal{M}$, of all meagre sets, and $\mathcal{N}$, of all null sets; that is $J = \mathcal{M}, \mathcal{N}$. It is useful to dwell on the meanings of these definitions with respect to these ideals. For instance, for $\mathrm{add}(\mathcal{M})$ we are seeking a family of meagre sets such that their union does not lie in $\mathcal{M}$. Recall that the Cohen reals may be collected into $\aleph_0$ delta-systems



Any individual singleton set $\{\xi\}$ is meagre so the union $\{\xi\} \cup \mathcal{A}$ where $\mathcal{A}$ is some existing family of meagre sets will not produce a non-meagre set. However, the collection $\mathcal{A}' = \{\{\xi_1\}, \{\xi_2\}, ...\}$ is such that each $\{\xi_i\}$ is meagre but $\bigcup \mathcal{A}'$ is not. Thus $\text{add}(\mathcal{M}) \le |\mathcal{A}'|$.

Concerning $\text{cov}(J)$, we ask: how can a family of meagre sets cover $\mathbb{R}$? To answer this question let us return to the decomposition of the skeleton of the real line in to sets:

2.1.2 Theorem, meagre-null decomposition

The line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero. That is, $A \in \mathcal{N}$ and $B = \mathbb{R} - A \in \mathcal{M}$.

A	A null set	$\mu^*(A) = 0$
B	A meagre set	$\mu^*(B \cap [0,1]) = 1$

A proof of this theorem was given in the paper that precedes this: *On the Continuum*. Alternatively, see either Oxtoby [1980] or Bartoszynski and Judah [1995]. Let us recall the definitions of Cohen and amoeba reals that were adopted in *On the Continuum*.

2.1.3 Definition, Cohen and amoeba reals

A **Cohen real** is a transcendental number not belonging to the meagre set A.
An **amoeba real** is a transcendental number not belonging to the null set B.

2.1.4 Notation for the sets of Cohen and amoeba reals

The set of all Cohen reals is denoted $\mathbf{C}(\mathrm{V})$; the set of all amoeba reals is denoted $\mathbf{A}(\mathrm{V})$; here the symbol V represents some ground model to which the Cohen and/or amoeba reals have been added. In our case imagine that we are adding reals to the ideal of all

finite subsets, $\mathbf{Fin} \cong \mathbf{2}^{<\omega}$, though this is embedded in some ground model for ZFC, denoted V.

A Cohen real is a transcendental number not belonging to any meagre set in **Fin**. To obtain a cover of meagre sets, we may start with the set B as this set embraces all the measure of the unit interval, $\mu(B)=1$ and contains all the extension points, so we need to adjoin to it all the boundary points, that is, the Cohen reals and the algebraic numbers. At the least we can do this by adding to the family continuum many singleton sets, $\{\xi\}$ where ξ is either Cohen or algebraic.

Concerning $\mathrm{non}(J)$, $J=\mathcal{M},\mathcal{N}$, let $\mathrm{NON}(J)$ be the family of all $x \subseteq \mathbb{R}$ such that $x \notin J$. Then $\mathrm{non}(J)$ is the cardinality of the smallest member of $\mathrm{NON}(J)$. In the meagre-null decomposition $A \in \mathrm{NON}(\mathcal{M})$ and $B \in \mathrm{NON}(\mathcal{N})$; thus $\mathrm{non}(\mathcal{M}) \le \mathfrak{c}$ and $\mathrm{non}(\mathcal{N}) \le \mathfrak{c}$. For $\mathrm{cof}(J)$ we must consider a basis for an ideal.

2.1.5 Defintion, basis

$\mathcal{A} \subseteq J$ is a basis for J iff $\mathcal{A}$ is cofinal in *J*: $(\forall x \in J)(\exists a \in \mathcal{A})(x \subseteq a)$.

Let $\mathrm{COF}(J)$ be the family of all bases for *J*. Any such basis is bounded above by $\mathfrak{c}$, since there cannot be more than continuum many members of any ideal. However, the size of $\mathrm{cof}(J)$ is indicated more by the following result.

2.1.6 Result on cardinal invariants

$\mathrm{non}(J) \le \mathrm{cof}(J)$

<u>Proof</u>

Note that $X \in \mathrm{COF}(J) \Rightarrow X \subseteq J$ is a basis for *J*. Let $X \subseteq J$ be a basis for *J* of minimal size. For every set $x \in X$ let $y_x = \omega - x$. Then the set $Y = \{y_x : x \in X\}$ is not covered by any member of *X*. Hence $Y \notin J$. So we have:, $X \in \mathrm{COF}(J) \Leftrightarrow Y \notin J$.

Since $|X| =$ the cardinality of the smallest member of $\mathrm{cof}(J) = \mathrm{cof}(J)$, we have : -

$\mathrm{non}(J) \le \mathrm{cof}(J)$.

This result is illustrative of the basic inequalities that obtain between these cardinal coefficients, as shown in the following diagram: -

$$\begin{tikzcd}
 & & \mathrm{cov}(J) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \mathrm{add}(J) \arrow[ur] \arrow[dr] & & \mathrm{cof}(J) \arrow[r] & 2^{\aleph_0} \\
 & & \mathrm{non}(J) \arrow[ur] & &
\end{tikzcd}$$

It has been shown that we may further relate the two ideals: $\mathcal{M}$, of all meagre sets, and $\mathcal{N}$, of all null sets. One important result is Rotherberger's theorem.

2.1.7 Rothberger's theorem

$\mathrm{cov}(\mathcal{M}) \le \mathrm{non}(\mathcal{N})$
$\mathrm{cov}(\mathcal{N}) \le \mathrm{non}(\mathcal{M})$

<u>Proof</u>

In the meagre-null decomposition we have sets: $A \in \mathcal{N},\ B \in \mathcal{M},\ A \cup B = \mathbb{R}$.

Take the quotient algebra, $\frac{\mathbb{R}}{A}$. The cosets are $b + A,\ b \in B$. We have $\frac{\mathbb{R}}{A} = B + A = \bigcup_{b \in B} b + A = \mathbb{R}$.

So $B + A$ is a cover for $\mathbb{R}$. Thus, $B \notin \mathcal{N} \Rightarrow B + A$ covers $\mathbb{R}$. In this case $|B + A| = |B| = \tau$. We have $B \in \mathrm{NON}(\mathcal{N})$, $B + A \in \mathrm{COV}(\mathcal{M})$. Furthermore, $|B + A|$ is an upper bound for $\mathrm{cov}(\mathcal{M})$ and $|B|$ is an upper bound for $\mathrm{non}(\mathcal{N})$. Now possibly there are some $X, Y \subseteq \mathbb{R}$ such that $|X| \le |A|,\ X \in \mathcal{N},\ X \cup Y = \mathbb{R}$. Then $\mathrm{cov}(\mathcal{M}) \le \tau$ and $\left|\frac{\mathbb{R}}{X}\right| = |Y| \ge \tau$. Hence $\mathrm{cov}(\mathcal{M}) \le \mathrm{non}(\mathcal{N})$.

The other result is proven similarly.

We use this theorem of illustrative of the process that may establish relations among the cardinal invariants we have so far introduced. There are other cardinal invariants. Two of these are: -

2.1.8 Bounding and dominating reals

$\mathfrak{b}$ - the size of the smallest unbounded family in ω^ω.

$\mathfrak{d}$ - the size of the smallest dominating family in ω^ω.

To clarify, we require a sequence of definitions: -

For $f, g \in \omega^\omega$, we have $f \le {}^*g$ if $f(n) \le g(n)$ for all but finitely many $n \in \omega$. Then: -

$\mathfrak{b} = \min\left\{|F| : F \subseteq \omega^\omega \text{ and } (\forall g \in \omega^\omega)(\exists f \in F)(f \not\le {}^*g)\right\}$.

$\mathfrak{d} = \min\left\{|F| : F \subseteq \omega^\omega \text{ and } (\forall g \in \omega^\omega)(\exists f \in F)(g \le {}^*f)\right\}$.

Note that $f \not\leq {}^*g$ resolves to $f > {}^*g$ and this automatically gives $\mathfrak{b} \leq \mathfrak{d}$.

Another important tool to compare cardinal invariants are Tukey functions: A **Tukey function** is an order preserving map that sends unbounded reals to unbounded reals; such a relation is denoted $\preceq_T$. Pawlikowski has shown that $\mathcal{M} \preceq_T \mathcal{N}$. (There is a proof in Tomek and Judah [1995].) This, some other observations on $\mathfrak{b}$ and $\mathfrak{d}$ enable us to construct the Cichón diagram: -

2.1.9 Cichón diagram

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uuu]
\end{tikzcd}$$

The arrows represent inequalities.

This provides considerable insight into the substructure of the Cantor set. However, there are yet more cardinal invariants:

2.1.10 A brief summary of other cardinal invariants

$\mathfrak{p}$	relates to the strong finite intersection property and pseudo-intersections
$\mathfrak{h}$	relates to maximal almost disjoint families and their shattering number
$\mathfrak{par}$	partition number relating to 2-colourings
$\mathfrak{s}$	splitting number - smallest cardinality of any splitting family
$\mathfrak{r}$	reaping number
$\mathfrak{hom}$	homogeneity number relating to 2-colourings
$\mathfrak{a}$	almost disjoint number
$\mathfrak{i}$	independence number

These are described in Halbeisen [2012]. Bartoszynski and Judah [1995] discuss some of these and also introduce yet others. In conclusion, to the casual observer the situation might be described as akin to a gothic nightmare. There are more cardinal invariants than most people have had hot dinners.

2.2 A wee sceptical issue

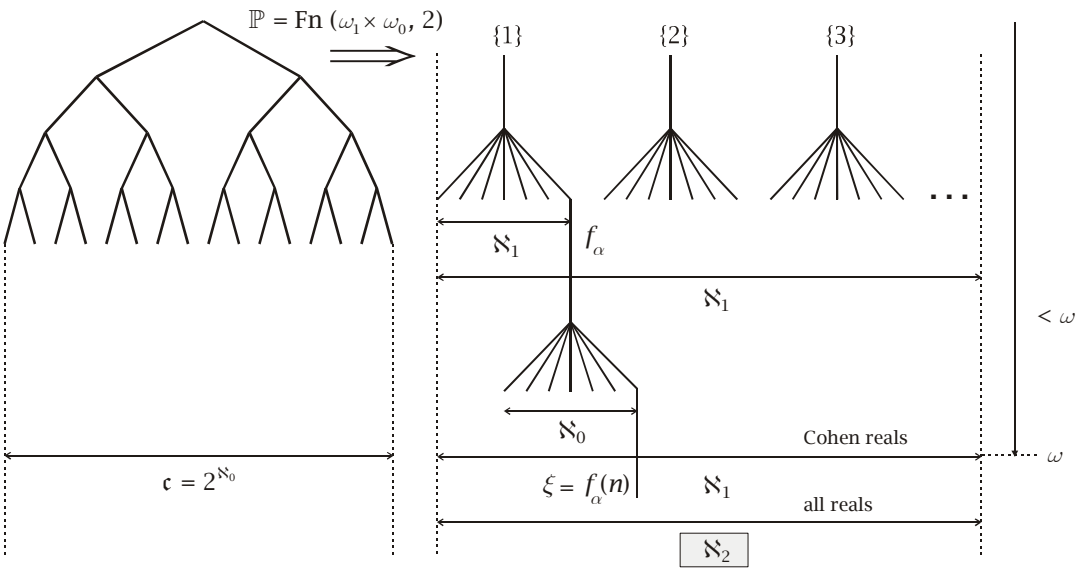
In the course of clarifying the definitions of these cardinal coefficients we have referred to singleton sets, $\{\xi\}$, where ξ is a real. In the Derived set ξ is itself a set and corresponds to an ultrafilter of **Fin** or alternatively to a subset $x \subseteq \omega$; ξ is constructed in the Cantor tree of boundaries as one of its actually infinite branches. However, the singleton set $\{\xi\}$ is not constructed by the tree as such. Within ZFC $\{\xi\}$ is constructed iteratively, but "outside" the Derived set. In order to construct a cover comprising meagre sets we must populate the boundary of the Cantor set with other sets not intrinsic to it. This is a subtle point, but one that indicates the somewhat arbitrary nature of set theory with respect to the continuum. Rather than adding axioms to ZFC perhaps we should consider subtracting something. If we disallow these singleton sets, then we may obtain the following definition of a Cohen real: a transcendental number not belonging to any meagre set of the continuum, and the notion of cover becomes ill-defined.

It might be a good idea to remind oneself of the remarks of von Neumann [1925] in his essay, *The Axiomatisation of Set Theory.* Von Neumann tells us that Zermelo, Fraenkel, Schoenflies used the axiomatic method to avoid the antinomies while keeping as much of "classical" mathematics intact as possible. The approach was pragmatic: to get the work done! Of course, many people do believe that ZFC is founded on an intuitive "iterative" concept of sets (for example, Boolos [1971]), and there may be some truth in this. But it is also useful to remind ourselves that set theory grew out of the C19th endeavour to resolve problems concerning the definition of integrals and measure in relation to the continuum, and that Cantor crystalized this movement with his invention of ordinal and cardinal arithmetic; yet the true empirical nature of the arithmetical continuum has not been fully appreciated, even to this day. Aristotle could not find the actual infinite in the phenomenology, so he discounted it altogether. Aristotle was essentially right not to derive the arithmetical continuum from phenomenology, though intuition can still help us in the formation of our theories, particularly with regard to the primitive notion of extension. Thus the arithmetical continuum is a theoretical construct that will ultimately derive its truth from evidence. We should construct the theory to fit the evidence and there is no *a priori* need to adhere to any part of ZFC longer than it proves useful. Of course, I am not advocating in any sense that we drop ZFC; but merely suggesting that, while ZFC is the tool par excellence for constructing models of the arithmetical continuum, it serves that purpose rather than defines it.

2.3 Playing with the model

Allowing the singleton sets, $\{\xi\}$, to exist and form part of the algebra of the continuum, then we have an upper bound: $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{c} = 2^{\aleph_0}$. Let $\mathrm{COV}(\mathcal{M})$ be the family of all families, $\mathcal{A}$, such that $\mathcal{A} \subset \mathcal{M}$ is a cover for $\mathbb{R}$. Then as an example of the kind of question that arises: could there be

an $\mathcal{A} \subset \mathcal{M}$ such that $\aleph_0 < |\mathcal{A}| < 2^{\aleph_0}$? For definiteness, let us ask whether we can have a collection $\mathcal{A}_1$ such that $|\mathcal{A}_1| = \aleph_1$. Cohen forcing with $\mathbb{P} = \mathrm{Fn}(\omega_1 \times \omega, 2)$ transforms the Cantor set into a structure where the Cohen reals are constructed at the ω_1 level, which introduces a meagre set of size $\aleph_1$.



In this diagram the collection $\{\{f_\alpha\} : \alpha \le \omega_1\}$ forms the required meagre set of cardinality $\aleph_1$. However, the forcing $\mathbb{P} = \mathrm{Fn}(\omega_1 \times \omega, 2)$ produces a model in which we have $\aleph_1 = |\mathbf{C}(\mathrm{V})| \le 2^{\aleph_0} = \mathfrak{c}$. If we assume that in V, the ground model, GCH holds, this still gives a model of CH: $2^{\aleph_0} = \aleph_1$. Thus, as the diagram also illustrates, in order to obtain a model of the continuum in which we have a distinct family of meagre sets, $\aleph_0 < |\mathcal{A}| < 2^{\aleph_0}$, we require a device for increasing the size of the continuum without increasing the size of the Cohen reals, $\mathbf{C}(\mathrm{V})$. This is where the introduction of partial orders corresponding to alternative forcings arise. The solution to this problem may be found in Bartoszynski and Haim [1995] (p.337). It proceeds by Sacks forcing.

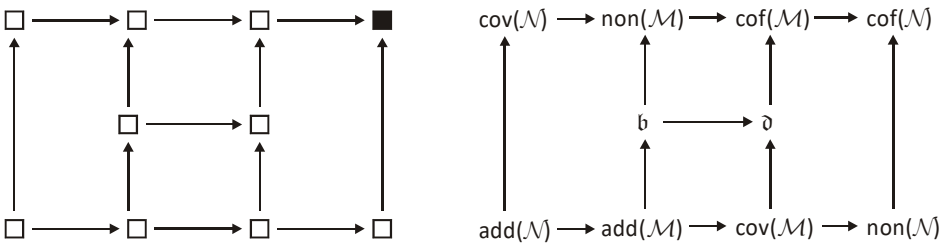
2.3.1 Definition, Sacks forcing

Let $\mathbb{S}$ be a partial order defined on the collection of all perfect subtrees of $\mathbf{2}^{<\omega}$ ordered by inclusion. Sacks forcing adds to the ground model any generic ultrafilter defined on $\mathbb{S}$.

Sacks forcing increases $2^{\aleph_0}$ without affecting the other cardinal coefficients. There are $\mathbf{2}^{\omega}$ perfect subtrees of $\mathbf{2}^{<\omega}$, and none of these are identified with the Cohen or amoeba reals per se. Hence, if we allow forcing with $\mathbb{S}$ we automatically increase the value of $\mathbf{2}^{\omega}$ relative to the other reals. If

the cardinality of the Cohen and amoeba reals taken together is $\aleph_1$ this gives a model in which $\mathrm{cov}(\mathcal{M}) = \mathbf{C}(\mathrm{V}) = \aleph_1$ and $\mathfrak{c} = \aleph_2 = 2^{\aleph_0}$.

Let $\square$ and $\blacksquare$ be two cardinals such that $\square < \blacksquare = 2^{\aleph_0}$. For example, $\aleph_1 = \square < \blacksquare = \aleph_2 = 2^{\aleph_0}$. In the absence of any further forcing notions, this gives a model of the continuum which may be illustrated by a Cichón diagram as follows: -

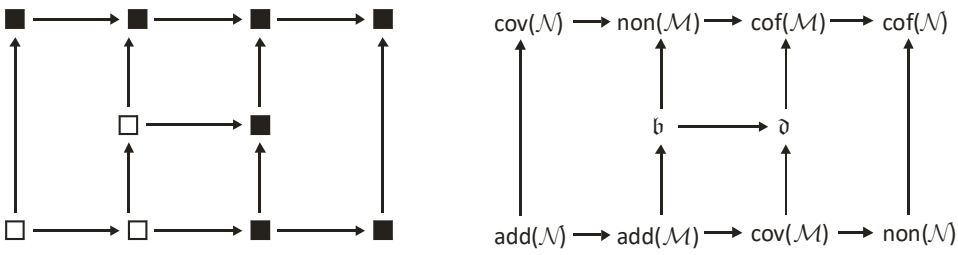


This really opens up the whole situation to the possibility of giving almost any fine structure we please to a model of the continuum. If the coefficient of $\mathfrak{c}$ may be increased relative to the other coefficients, the question arises: what possibilities are there for adjusting the value of any given coefficient relative to others, subject only to the constraints provided by the Cichón diagram?

It is not the purpose of this monograph to take this matter further, especially as the project has been extensively researched and there is a description of the results in the works of Bartoszynski and Judah [1995] and Halbeisen, Lorenz J. [2012] and others. The situation may be illustrated by Miller's model *A*, described in Bartoszynski and Judah [1995].

2.3.2 Model A (Miller 1981)

In Miller's A model the cardinal invariants take the following values: -



One way to generate this model involves a type of countable support iteration with a partial order: $\mathbb{P}_\alpha : \alpha < \omega_2$ that alternates between a kind of forcing called random (See definition 2.4.2 below) and Cohen forcing: -

The forcing is random, **B**, if α is even

The forcing is Cohen, **C**, if α is odd.

This also gives meaning to the term *countable support iteration.* There is a single partial order, $\mathbb{P}_\alpha : \alpha < \omega_2$, but within this we may identify two countably infinite sub-sequences: one adding random reals and the other adding Cohen reals.

1. It is a result (Bartoszynski and Judah [1995]) that Cohen forcing adds unbounded (and splitting) reals but not dominating reals. Hence, the countable Cohen forcing results in $\mathfrak{b} = \aleph_1$.
2. It is a result (Bartoszynski and Judah [1995]) that Random forcing adds no unbounded reals; that is, does not increase the size of the cardinal invariant, $\mathfrak{b}$. Hence in this model we retain $\mathfrak{b} = \aleph_1$. However, random forcing does add dominating reals, so this means $\mathfrak{d} > \mathfrak{b}$.
3. It is a result (Bartoszynski and Judah [1995]) that Cohen forcing entails $\text{non}(\mathcal{N}) = 2^{\aleph_0}$. As we take the ground model to be a model of CH (or GCH), this means that, in the absence of any other forcing, $2^{\aleph_0} = \aleph_2$.
4. Unlike the preceding model we do not have Sacks forcing to bump up the value of the continuum at the last stage, so $2^{\aleph_0} = \aleph_2$ remains unaltered.
5. This model is akin to the one preceding it, with $\mathbb{P} = \text{Fn}(\omega_2 \times \omega, 2)$, which gives an overall value of $2^{\aleph_0} = \aleph_2$ for the continuum, but does not permit the construction of a cover such that $\text{cov}(\mathcal{M}) = \aleph_1 < \aleph_2 = 2^{\aleph_0}$. In this respect we may refer to a result of Stern cited in Halbeisen [2012] (p. 374): "If G is $\mathbb{C}_{\omega_2}$ over V where $\text{V} \vDash \text{GCH}$ then in $\text{V}[\text{G}]$ there is no partition of ${}^{\omega}\omega$ into ω_1 disjoint closed sets."

2.4 Concerning the distinction between random and amoeba forcing

The model that we have described above is very similar to the one we gave subject to the Axiom of Completeness in which we have: -

2.4.1 Model of the continuum subject to the Axiom of Completeness

1. CH: $2^{\aleph_0} = \aleph_1$
2. Cohen forcing: $\mathbb{C} = \text{Fn}(\omega, 2)$.
3. Amoeba forcing.
4. All the cardinal invariants = $\aleph_1$.

Some explanation of how the Axiom of Completeness model can be different from Miller's A is needed. The key is that what is called random* forcing in Bartoszynski and Judah [1995] is not the same as amoeba forcing. Random forcing within the context of first order set theory (ZFC) is defined relative to an encoding of sequences that converge on random reals. These sequences are called Borel codes.

2.4.2 Borel codes

Every Borel subset of the continuum can be constructed from basic sets in countably many steps. The information about this construction can be stored in one real number. BOREL denotes the set of Borel codes.

Random forcing is defined specifically as: -

2.4.3 Random forcing

$\mathbf{B}(\kappa) = \{[A]_{\mathcal{N}} : A \in \mathrm{BOREL}(2^{\kappa})\}$ where $\mathcal{N}$ is the null ideal.

It is relative to this definition that Miller's A model is constructed with a partial order $\mathbb{P}_{\alpha} : \alpha < \omega_2$ and the results that support that conclusion, described so ably by Bartoszynski and Judah [1995] are derived.

2.4.4 Amoeba forcing

$\mathbf{A}_{\varepsilon} = \{U : U \subseteq \mathbf{2}^{\omega}, U \text{ is open and } \mu(U) < \varepsilon\}$.

Amoeba forcing is not forcing in the sense in which random forcing with BOREL codes is, or Cohen forcing. Given Cohen forcing, $\mathbb{C} = \mathrm{Fn}(\omega, 2)$, amoeba forcing is what is left over in the continuum once the Cohen reals and other boundary points are removed from it.

2.5 A playful universe of sets

What set theorists have achieved is a playful universe of sets. It is a universe in which there are seemingly unbounded possibilities for the fine structure of the continuum. The situation for $2^{\aleph_0} = \aleph_2$ is particularly well understood; every possible fine structure consistent with the Cichón diagram and other relations between those cardinal invariants that do drop out of ZFC is possible.

However, what is now lacking is any motivation to make alterations to our understanding of the continuum. Up until now, what has been missing in the discussion is a proper appreciation of the centrality of the Axiom of Completeness – how it provides a simple and unambiguous

structure to the continuum, and one that clearly establishes CH, $2^{\aleph_0} = \aleph_1$, as the ground model or starting point for any refinements.

Another possible criticism of the proliferation of fine structure models of the continuum is that these have not been developed with regard to the continuum itself, and its intrinsic properties. Specifically, the absence of a second primitive notion, that of extension, needed to explain precisely why space is space and not a mere collection of points, has not been appreciated. Amoeba forcing captures this notion – in that it expresses the idea that when the boundary points have been removed from space, nothing has been taken away and all of space still remains. I suggest that it is unlikely that any model of the continuum in which amoeba forcing is neglected will succeed as a candidate. Yet amoeba forcing appears to be the least understood or charted species of forcing of all those available. Nonetheless, the emergent situation is very exciting; the prospect of a re-invigorated empirical study of the infinitesimal structure of space is presented.

3. Motivation for Change

3.1 Non-measurable sets

We may prove that, subject to the Axiom of Completeness, there are no measurable sets.

3.1.1 Definitions, *F, G* sets

The F_σ sets are countable unions of closed sets.

The G_δ sets are countable intersections of open sets.

3.1.2 Definition, measurable

A set A is **measurable** (Lebesgue measurable) if for each $\varepsilon > 0$ there exists a closed set F and an open set G such that $F \subset A \subset G$ and $m^*(G - F) < \varepsilon$.

3.1.3 Theorem [Vitali]

There exists a set of real numbers that is not Lebesgue measurable.

<u>Proof</u>

Consider the quotient structure, $\Phi = \frac{[0,1)}{\mathbb{Q}}$. The cosets are $\mathbb{Q} + \xi, \xi \in \mathbb{R}$. Choose a representative from each coset, $\tau_\alpha \in \mathbb{Q} + \xi_\alpha$. Let $M = \{\tau_1, \tau_2, \ldots\}$. Make all translation

invariant copies of *M*, denoted $M_r = M + r,\ r \in \mathbb{Q} \subset [0,1)$. Suppose M is measurable, then $\mu(M) \neq 0$ and $\mu(\Phi) = \mu(M) = \mu(M_r) = \mu([0,1)) = 1$. But at the same time we have: -

$$\bigcup M_r = \bigcup \{M_r : r \in \mathbb{Q},\ 0 \le r < 1\} = [0,1) \qquad \textbf{(1)}$$

Countable subadditivity is $\mu\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mu(I_k)$. Applying this to **(1)** we obtain: -

$1 = \mu([0,1)) = \mu\left(\bigcup M_r\right) = \sum \mu(M_r) = \sum 1 = \infty$. A contradiction. Hence *M* is not measurable.

The principles on which this argument by contradiction rest are: -

(1) That every set is measurable.

(2) Countable additivity: $\mu\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mu(I_k)$.

(3) That measure is translation invariant: $\mu(X) = \mu(X + a),\ a \in \mathbb{R}$.

(4) Axiom of choice to construct *M* by arbitrarily selecting a representative from a coset of cardinality $\mathfrak{c}$.

If we are not prepared to give up (2), (3) and (4) then we must abandon (1). Principles (2), (3) and (4) follow from the Axiom of Completeness.

3.2 Why there are non-measurable sets

In essence the existence of non-measurable sets derives from the existence of extension points in the continuum. It is the role of an extension point to fill up space, where this property *of being extended in space*, is a primitive notion deriving initially from intuition. In the phenomenological, intuitive understanding of space we literally perceive lines as extended in space; we transfer this property to our empirical concept of the arithmetical continuum. However, as there are cardinally as many points in any interval whatsoever, and indeed, in any space of any dimension (owing to the existence of space-filling curves) it follows that there is an ambiguous element to the notion of measure. Under the Axiom of Completeness we cannot derive the difference in the measure of one interval from another contained within it from their cardinality. So there is no function in the mathematical sense from cardinality to measure. At least, not under the assumption of CH.

Thus, it is the amoeba reals corresponding to the *S* numbers of the Mahler classification of transcendental numbers that are essentially non-measurable. The entire set of *S* numbers make up the measure of any interval; there are continuum many, $\mathfrak{c}$, in any interval. Thus, $[0,1] \div \mathfrak{c}$ has no meaning. Let *ds* represent an arbitrary amoeba real, an *S* number. Then $\mu(ds) \neq 0$, for

otherwise it would be a boundary point. On the other hand we cannot give a determinate measure to $\mu(ds)$, say $\mu(ds)=1$, for otherwise, there being $\mathfrak{c}$ many of them, the measure of any interval would be infinite. So, under the Axiom of Completeness, there must be non-measurable sets.

To confirm this, let us examine in detail the fine structure of an S number and compare this with a measurable set.

Firstly, it is not the intentiion here to describe in detail the origin, definition and properties of measure. For these, see Bessound [2008]. To keep things simple, we state merely that a measure μ is such that the measure of any interval I is equal to the length of that interval, as one would intuitively determine it: $\mu(a,b)=\mu[a,b]=b-a$.

3.2.1 Definition, measurable

A set A is measurable iff

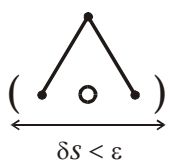
$(\forall\varepsilon>0)(\exists F)(\exists G)(F$ is closed, G is open, $F\subset A\subset G$ and $\mu(G-F)<\varepsilon)$.

To put this into layman's terms: an "individually" measurable set is one that has a boundary of zero measure (a null set), the boundary being the difference between its "outside" and its "interior". Diagrammatically, a measurable set looks like the following.

$G \quad F$

$(\; [\qquad] \;)$

A measurable set could be a countable union of such sets; hence we should extend the definition to F_σ and G_δ sets. It is a result that a set is measurable iff it can be represented as an F_σ plus a null set or as a G_δ minus a null set. Taking the diagram above as canonical of the representation of a measurable set, let us compare Cohen with amoeba reals (Mahler S numbers). Firstly, concerning the Cohen real (the U, T numbers of the Mahler classification), recall that such a number has diagrammatic representation: -

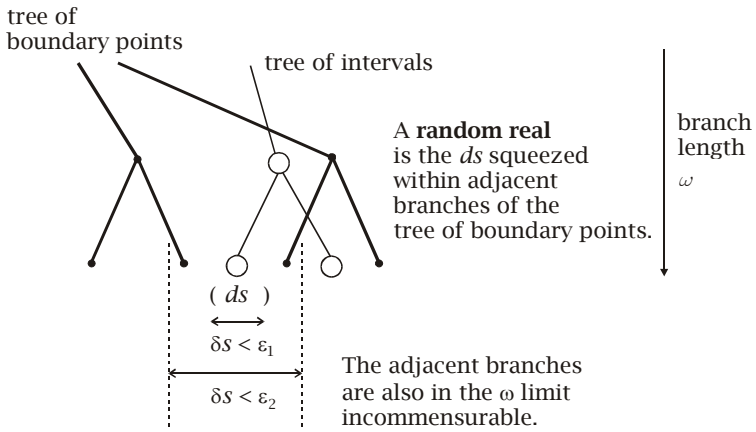


Furthermore, this is the terminal ωth node of a branch in the Cantor set; the distance ε is incommensurable with 0 and the two branches are "identified". They are enclosed within an open set of measure 0. Hence a Cohen real has set theoretic representation: -

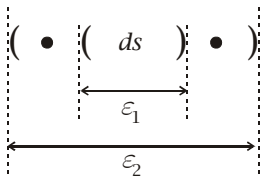
$G \quad F$

$(\quad \{\cdot\} \quad)$

where $\mu(G-F)<\varepsilon$, G is open and F is closed. So clearly a Cohen real is measurable. On the other hand an amoeba real (Maher S number) has representation: -



That is: -



So we see clearly that an amoeba real is non-measurable. Another way to consider this point is by means of Ulam's theorem: -

3.2.2 Ulam's Theorem

A finite measure μ defined for all subsets of a set X of power $\aleph_1$ vanishes identically if it is equal to zero for every one-element subset. The meaning of identically zero is: $\mu(X)=0$.

<u>Proof</u>

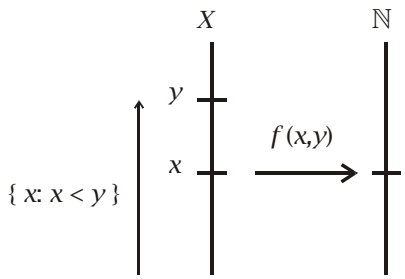
(derived from Oxtoby [1980])

$|X|-\aleph_1$

Hence, X can be well-ordered so that $(\forall y \in X)(\{x : x<y\}$ is countable$)$.

This in turn means that there exists a function, f, such that $f(x,y)$ is a mapping of $\{x : x<y\}$ onto $\mathbb{N}$. Write this as: -

$$f(x,y)\begin{cases} X^2 \to \mathbb{N} \\ (x,y), x,y \in X \to f(x,y) = n \end{cases}$$

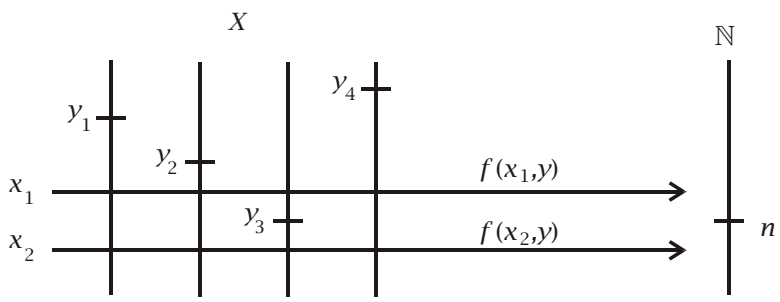


This is one-one below any given y. That is,

$x < x' < y \Rightarrow f(x,y) \neq f(x',y)$.

Then we can define sets as inverse images of this mapping subject to the condition that they map to the same $n \in \mathbb{N}$:

$\mathrm{F}_x^n = \{y : x < y, f(x,y) = n\}$



In this diagram we have $\mathrm{F}_{x_1}^n = \{y_1, y_2, y_4\}$ and $\mathrm{F}_{x_2}^n = \{y_1, y_2, y_3, y_4\}$.

This collection of sets can be represented as an array with $\aleph_0$ rows and $\aleph_1$ columns: -

$$\begin{array}{ccccc} \mathrm{F}_{x_1}^1 & \mathrm{F}_{x_2}^1 & \dots & \mathrm{F}_x^1 & \dots \\ \mathrm{F}_{x_1}^2 & \mathrm{F}_{x_2}^2 & \dots & \mathrm{F}_x^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathrm{F}_{x_1}^n & \mathrm{F}_{x_2}^n & \dots & \mathrm{F}_x^n & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

This matrix has two properties:

(1) Sets in any row are mutually disjoint.

(2) The union of any sets in a column equal X less a countable set, given specifically by $\{y : y \leq x\}$.

(For proofs of these assertions see Oxtoby [1980].)

From (1) it follows that there are at most countable many sets in any row such that $\mu\left(\mathrm{F}_x^n\right) > 0$ as $\mu(X)$ is finite. This also entails that there are only countably many such sets in the whole array.

From (2): Since there are only countably many columns, $(\exists x \in X)\left(\mu\left(\mathrm{F}_x^n\right) = 0\right)$ for all *n*.

The union of the sets in this column has measure zero. Its complement is also a set of countable many sets and has measure zero. This entails $\mu(X) = 0$. That is, μ is identically zero.

3.2.3 Generalisation of Ulam's theorem

Ulam's theorerm applies to all X such that $\mathrm{card}(X) = \aleph_n$. In other words, only if the cardinality of X is weakly inaccessible or greater does the theorem fail.

Assuming the continuum hypothesis (or something like $2^{\aleph_0} = \aleph_n$) we get the following theorem: -

3.2.4 Proposition

A finite measure defined for all subsets of a set of power *c* vanishes identically if it is zero for points.

So Ulam's theorem confirms that, subject to conditions, we cannot ascribe a definite non-zero measure to all points of the continuum, and concretely we have shown that the amoeba, Mahler *S* numbers have no measure. The minimum condition is that the continuum satisfies $\mathbf{c} = 2^{\aleph_0} = \kappa$ where $\kappa <$ the first weakly inaccessible cardinal. Hence the possibility arises that if we allow the cardinality of the continuum to be greater than the first weakly inaccessible, then every set of reals may become measurable. Indeed, Solvay in a paper published in 1970 entitled *A model of set theory in which every set of reals is Lebesgue measurable* has demonstrated precisely this possibility.

3.3 The Poincaré recurrence theorem: A way forward?

It is beyond the scope of this paper to "solve" the problem of how to empirically test a model of the continuum. I will conclude with just a suggestion of one possible avenue to explore in this respect. Given the importance of measure to functional analysis, and that function spaces akin to the space of the continuum are essential parts of the solutions to differential equations, there must be many more.

The source of the definitions that follow is Oxtoby [1980]. Physical systems, such as the solar system, appear to be stable. Planets move in orbits: after a given period of time a planet will

return to the same state. This is a manifestation of the conservation of energy. Poincaré's recurrence theorem states necessary conditions for recurrence to occur.

3.3.1 Recurrence

Let X be a bounded open region of r-space, and let T be a homeomorphism of X onto itself that preserves volume: that is, G and $T(G)$ have equal volume, for every open set $G \subset X$. Under iteration of T, each point x generates a sequence:

$x, Tx, T^2x, T^3x, \ldots$

Which is called the *positive semiorbit* of x. A point x is said to be *recurrent with respect to* G if $T^i x$ belongs to G for infinitely many positive integers i.

3.3.2 Definitions, *S*-measurable, wandering, dissipative

Let X be a set and let S be a σ-ring of subsets of X, and let I be a σ-ideal in S. A mapping T of X into X is called *S-measurable* if $T^{-1}E \in S$ whenever $E \in S$. A set $E \subset X$ is called a *wandering set* if the sets $E, T^{-1}E, T^{-2}E, \ldots$ are mutually disjoint. T is called *dissipative* if there exists a wandering set that belongs to $S - I$; otherwise, T is called *nondissipative*. For any set $E \subset X$, let $D(X)$ denote the set of points x in E such that $T^i x \in E$ for at most a finite number of positive integers i. T is said to have the *recurrence property* if $D(E) \in I$ for every $E \in S$.

3.3.3 Theorem

An S-measurable mapping T of X into X has the recurrence property iff T is nondissipative.

Oxtoby comments, "This means that almost all points of any measurable set E return to E infinitely often under iteration of T. In particular, for any open set $G \subset X$, all points of G except a set of measure zero are recurrent with respect to G."

3.3.4 Definition, recurrent

A point x is said to be *recurrent under* T if it is recurrent with respect to every neighbourhood of itself. Such a point was called by Poisson "stable à la Poisson."

3.3.5 Poincaré Recurrence Theorem

If T is a measure-preserving homeomorphism of a bounded open region X of r-space onto itself, then all points of X except a set of first category and measure zero are recurrent under T.

I observe that the recurrence theorem requires that the transformation in question be "measure-preserving" and Oxtoby's gloss refers to "any measurable set". Therefore, I suggest that exposition of the physical stability of the universe may depend upon the fine, infinitesimal structure of the continuum. If Poincaré's Recurrence Theorem requires that all sets be measurable, then we may be obliged by nature to adopt a model of the continuum in which all sets are measurable, and the stability of the universe may empirically confirm this choice. The development of this suggestion lies outside the scope of this paper; it is just one avenue in which research might progress. Recognising that the theory of the arithmetical continuum belongs to the empirical domain should not cause us to despair of an ultimate solution to the determination of its fine structure.

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