Answer to Hamming

Reply to Hamming's essay

Mathematics on a Distant Planet

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Abstract

This essay is a refutation of the techno-realist, positivist philosophy of mathematics advanced by R.W. Hamming in his paper *Mathematics on a Distant Planet* (Published by the American Mathematical Monthly. Vol. 105. No.7). A Platonist reply defends the view that mathematical knowledge gives rise to a transcendental deduction whose conclusion is that the mind is not a material entity. The paper reviews the cultural dominance of Positivism in contemporary philosophy, arguing that on a just estimation of the arguments the dialectical battle between the Gods and Giants is more even than appearances would allow. Ostrich nominalism and the formalist philosophy of mathematics are rejected. The transcendental deduction goes through because of the ontological commitment involved in quantification in mathematical statements. Reviewing the analysis of that commitment in the work of W.V.O. Quine, not only is there ontological commitment in first-order set theory, the paradigm of a Platonist theory, but both number theory and analysis, via the Axiom of Completeness, are irreducibly second-order theories, for which the nominalist's myth advanced by Quine cannot succeed. The cultural and spiritual significance of the debate is examined: the belief in universals and/or abstract objects is a ground for rational belief in immortality. In an appendix it is argued that limits cannot be constructed on the basis of the potential infinite alone.

Answer to Hamming

Hamming's essay *Mathematics on a Distant Planet*[1] is the battle cry of the "techno-realist"[2]. It is a paper famous for a number of challenging statements – particularly for Hamming's condemnation of the Lebesque integral: "Indeed, for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue but not Riemann integrable, then I would not fly in it." Equally for his appraisal of set theory and its father: "I am inclined to believe that they would have confined Cantor in his old age to an insane asylum." This is particularly vitriolic, given that Cantor did not reach old age because he committed suicide, and according to popular belief because of the lack of recognition of his work, led by Kronecker. The essay is also notable for Hamming's method of assessing theses: "Many years ago, as I was picking up a paper to read on the non-computable numbers, I suddenly realized that no one could ever come into my office and ask for a non-computable number. If they never can occur, why bother? So I threw the article in the waste basket unread!"

The prejudice behind these statements does not seem to justify the emotive appeal that the article seems to still generate, and all the more so on account of the ready observation that with the possible exception of Hamming's assertion that the only numbers to exist are computable numbers, there is no systematic philosophy of mathematics in the article at all. For example, there is no presentation of an epistemology of mathematics, and he hovers uncertainly between a form of strict finitism and a commitment to the Aristotelian notion of the potential infinite, possibly conflating the two points of view. He rejects the notion that a proof has bearing on the truth of a theorem, but offers no alternative criterion. He argues against proof by contradiction on the grounds that such proofs involve self-reference, but not wishing to reject all such proofs, commentsvaguely that, "It is a question of how much self-reference is acceptable in a self-reference proof." So the reputation of the paper for greatness is suspect.

Mathematics on a Distant Planet is the work of a man who is philosophically naive. It is naive because Hamming is either is blind or chooses to be blind to the nature of philosophy, which is a dialectical process in which for every thesis there is an antithesis. For after all, every philosopher is arguing against someone else, even if that "other" is nothing more than a straw-man. The activity of philosophy implies the right to reply. Genre wise, *Mathematics on a Distant Planet* is a monologue, and a very specific kind of monologue, a sermon.

Since its popularity does not rest on its internal merits, the success of the paper needs a socio-historical explanation. One of the aim of this paper is to provide that explanation; the other is to provide an assessment of the extent to which the philosophy against which Hamming is arguing has been decisively refuted.

The philosophy of mathematics that Hamming is rejecting goes broadly under the banner of Platonism. That Plato is the man against which he is arguing turns up explicitly in Hamming's paper towards the end: "In the Platonic world that most mathematicians seem to think mathematics is, one "discovers" theorems that apparently were already there immediately after the big bang." His argument against the undefined position of Platonism is merely that "Mathematicians on this Earth generally realize that the Platonic view is not defensible logically...". There is no explanation as to why the Platonic view is not defensible and, given his attack on logic generally, the addition "logically" is particularly vague. This is a fallacious argument from authority: Hamming knows it is not "logically defensible", therefore it is not.

So has Platonism been decisively refuted? Have we reached a time in the early C21st at which no rational person could even remotely defend any form of Platonism?

The late Gertrude Anscombe in her seminars was fond of saying, "Hume is not a package-deal". The same dictum could be applied to Plato – For Plato is not a package deal. The following parts of the "Plato package" may be deleted without affecting his philosophy of mathematics: Plato believed in a demi-urge who fashioned the world in the likeness of timeless forms; he believed that all knowledge depends on the recollection of prior communion with these timeless forms in a pre-existence, from which we fell – in the religious sense of fall – and hence, he took the existence of these forms as evidence of metempsychosis. His preference for government by philosopher-kings who have privileged access to this world of forms is also well-known. Though these beliefs are not popular in Western philosophy faculties they might even now be espoused by some, and do crop up in modern variants of Eastern religions – particularly in sects of Hinduism and Buddhism. The influence of Plato on the early Christian fathers can also not be denied – recall the term *logos* in the Gospel according to St. John: "When all things began, the Word already was. The World dwelt with God, and what God was, the Word was."[3] The theological importance of the Plato package to Christianity is indisputable.

However, the core of Plato's philosophy of mathematics is the argument against the empirical philosophy of his times that is attributed by Plato to leading Sophists, particularly Protagoras; this argument pervades the whole of his work, but can be seen explicitly in the dialogues *Theatetus* and *Sophist*. Good to start the exposition of this argument with a description of the opposed viewpoint – nominalism – and where better to find an explanation of this than the classic statement of Hobbes?

> Of Names, some are *Proper*, and singular to one onely thing; as *Peter*, *John*, *This man, this Tree*: and some are *Common* to many things; as *Man*, *Horse*, *Tree*; every of which though by one Name, is nevertheless the name of divers particular things; in respect of all which together, it is called an *Universall*; there being nothing in the world Universal but Names; for the things named, are every one of them Individuall and Singular.
>
> One Universall name is imposed on many things, for their similitude in some quality, or other accident: And whereas a Proper Name bringeth to mind one thing onely; Universals recall any one of those many.[4]

In this passage Hobbes denies the existence of universals, claiming that the only things that exist are individuals; a modern materialist might take these individuals to be particular material entities located in space and time. However, Hobbes also appears to contradict himself when he writes of "similitude in some quality" (that is, property), so it appears that he has not thought through the doctrine entirely. A similar contradiction may be found in Hume, who disingenuously pretends never to have heard of Plato in the opening statements of his *Enquiries*. Having introduced his empirical theory of knowledge: "… all our ideas or more feeble perceptions are copies of our impressions or more lively ones"[5], he then ignores Plato when he throws down the challenge: "Those who would assert that this position is not universally true nor without exception, have only one, and that an easy method of refuting it; by producing that idea, which, in their opinion, is not derived from this source."[6] One wishes to reply: universals. Nonetheless, having denied even having heard of the Platonic doctrine of universals he contradicts himself when he discusses colour – using the specific example of shades of blue[7] - and this accident illustrates just how difficult it is to escape universals, once one begins to consider them.

It is in the *Theatetus*where Plato explicitly argues against the empiricist doctrine that "perception is knowledge".[8] Plato advances the doctrine that common-terms, which he equates with forms, are apprehended only by thought; knowledge of meanings cannot be abstracted directly from particular sense-experience.

It is true that the doctrine of universals is difficult to comprehend, though this depends to an extent on socio-historical context, for the whole of our Western system of education is a training in empirical and material thinking, so that when the notion of something *immaterial* crops up, one is literally untrained to deal with it. The terms used may also not help: property, accident, quality, universal and form. Notwithstanding this plethora of terminology, I also recommend that one thinks of universals as *structures*. When, for instance, I look at a blue object, I see not only the object, but its*structural similarity* to other blue objects; this structure is as much a part of my phenomenological experience of the world as the individual object is. Nor does it affect this argument to recognise that there is something conventional about what is called blue, and that, for instance, not every language even has a term for blue. The point is that one shade of blue is similar to another, even if not identical, and where the boundary lies between, say, blue and green, is irrelevant. I experience not only individuals, but also structures.

In keeping with the principle of deconstructing the Plato package, it is worth remarking that there is more than one way to treat of the real existence of universals. Plato's approach is to separate the universals into a realm of Forms; that requires that one postulates a special faculty of the mind for apprehending this realm (existence *ante re*); also there is the question of how the forms participate in the material world – the problem of participation that he raises in the *Parmenides*. However, one does not have to go this far. The universals need not exist separately from the individuals – this is Aristotle's alternative approach – and they need not be thought of as entities at all (existence *in re*). However, the question *how we know them* still remains.

Hence, it is now commonplace to distinguish a universal – that which can be instantiated – from an abstract entity – that which has neither spatial nor temporal location, and is causally inert. A "realist" is said to be someone who believes in universals; a "Platonist" someone who believes in abstract entities. The term "realism" here could be confusing; for the belief in a real world of physical entities is also called "realism", and that is how the term realism shall be used in this paper. Thus, I shall use the term "Platonism" for the belief in universals in this paper, taking that to the minimum part of the Plato-package for one to qualify as a Platonist.

The distinction is important, but the "problem of universals" remains, regardless of whether one believes in them *ante re* or *in re*. For, whatever they are, if we know them, then the question is *how do we know them?*

It is this question of how we know universals that leads us to a transcendental deduction[9]. If the mind is capable of knowing *structures* (forms, properties, universals), then *the mind is not a material entity*. Plato in the *Theatetus* makes this step when he argues that the power of the mind to make judgements about different types of sense-object, that is, to compare and contrast them, indicates that the mind itself is independent of the sense-organs.

> SOCR. Now take sound and colour. Have you not, to begin with, this thought which includes both at once – that the both *exist*?
> THEAET. I have.
> SOCR. ... Then through what organ do you think all this about them both? What is common to them both cannot be apprehended either through hearing or through sight. Besides, here is further evidence for my point. Suppose it were possible to inquire whether sound and

> colour were both brackish or not, no doubt you could tell me what faculty you would use – obviously not sight or hearing, but some other.[10]

From this position, Plato believes he has established that the mind (soul) is non-material, and he steps from there to the immortality of the soul. In contrast to what I have said about the Plato-package above, I maintain that a transcendental deduction does follow on from the doctrine of universals. It is *not an added extra of the Plato package, but lies at its very core*.(I do not maintain that this proves the immortality of the soul – that is an extra.) This doctrine distinguishes mind from matter in that the mind is possessed of a faculty for knowledge of meanings that cannot be abstracted from mere combinations of material atoms. That, in itself, is no more than another aspect of the "problem of consciousness"; a philosopher of materialistic inclination, one who is unable to be a strict nominalist, can still argue that all real, causal processes take place in the material substratum of reality – the world described by physics – while consciousness is an epiphenomenon that "supervenes" on this material substratum. Not possessed of any causal power itself, it is produced by brain states.[11]

Yet, at the minimum, acceptance of the existence of universals produces a model of the world that is in some way, even if not at the level of substance, split into two realms: (a) the realm of spatio-temporal entitles, subject to causal laws, and (b) the realm consciousness phenomena, wherein we experience awareness, understanding and ratiocination.

Now for the question: what is the relevance of all this to the philosophy of mathematics? Answer: mathematics is prima facie littered with nothing but universals, and of these, two particularly important candidates are *number* and *infinity*.

Yet with this answer, we raise yet again the spectre ofthe demon of metaphysics that Carnap and so many others sought to exorcise from philosophy. But before we go on to the description of that socio-historical development, let us continue to explore the Platonic vision.

I wish particularly to sketch into this exposition what Plato pictured in *The Sophist* as the battle between the Gods and the Giants, and which Cornford equates with the dialectical conflict between Idealists and Materialists[12].

> STRANGER. What we shall see is something like a Battle of Gods and Giants going on between them over their quarrel about reality. ... One party is trying to drag everything down to earth out of heaven and the unseen, literally grasping rocks and trees in their hands; for they lay hold upon every stock and stone and strenuously affirm that real existence belongs on to that which can be handled and offers resistance to the touch. They define reality as the same thing as body, and as soon as one of the opposite party asserts that anything without a body is real, they are utterly contemptuous and will not listen to another word. ... and, accordingly their adversaries are very wary in defending their position somewhere in the heights of the unseen, maintaining with all their force that true reality consists in certain intelligible and bodiless Forms. In the clash of argument they shatter and pulverise those bodies which their opponents wield, and what those others allege to be true reality they call, not real being, but a sort of moving process of becoming. On this issue an interminable battle is always going on between the two camps.[13]

This is a profound foretelling of the history of Western Philosophy that according to some only began with Plato: empiricists and materialists on the one hand – the Giants – in eternal war with the rationalists and idealists on the other – the Gods. Of course, each philosopher's work is a unique blend of doctrines that sometimes borrow a flavour from the other side, but any philosopher can be associated with one or other side of this argument, even if confusedly.

Following Cornford I above equated the Giants with the empiricists and materialists, but I need to clarify this point a little. Not every contemporary philosopher subscribes to materialism in the sense advanced by David Armstrong[14]. But I am using the term Giant in a broader sense of "one who is sympathetic to the doctrines advanced by Armstrong" and of "one who would expunge metaphysics from philosophy in the sense advocated by Rudolf Carnap". It is a catch-all for that kind of dogmatic rationalising spirit that pervades contemporary thought. For example, in this sense epiphenomenalists, while not technically materialists, belong to the School of Giants. For what is epiphenomenalism other than the doctrine espoused by a materialist who cannot quite stomach the idea that consciousness is identifiable with brain states? It is the underlying commitment to explanation by means of causally efficacious objects.It is in this sense that the Giants comprise a great School.

Thus to the socio-historical context. Broadly speaking, the C19th was the century of the Gods and the C20th, of which our contemporary scene is but an extension, has been the century of the Giants; between the two epochs a battle was fought in which the Giants won a tremendous victory over the Gods, the result of this battle was that the Gods were all but eradicated from Western philosophy faculties. My reader must know this, for he or she will quickly identify which side of the war he or she supports, and a quick calculation will tell one the general affiliation of every other member of faculty that he or she knows. That skirmishes exist between different champions of the Giants' cause cannot be in doubt, but that they all belong fundamentally to the same school of thought is also not in question. That is, unless there are any closet Platonists.

In this decisive battle the Giants were assisted by an overwhelming cultural development that was brought on by the second scientific revolution – the one which saw the triumph of theoretic physics, and produced the special theory of relativity, quantum mechanics, television, flight and the atom bomb. Until roughly 1900 most Westerners, at every level of society, were Christians, or at least, claimed they were. They accepted that in some sense the Bible was the revealed work of God, and believed that the world was in some sense created by Him. The work of the philosopher was to come after the fact of this culture, and provide justifications and glosses for what was essentially the religious framework of society. Then there was a huge revolution in cosmology; Western culture came to reject the creationist viewpoint and adopted instead an alternative account of the origin of Man and the Universe: the Cosmos is driven by material forces that produce either a one off Big Bang, or an endless cycle of bangs, possibly in a recurrent loop of time, and Man is the evolutionary product of this process. No place in this schema for concepts, meanings, all that is particular or unique to man, his personal psychodrama, his pilgrimage in life – his awareness, his consciousness – all must be subsumed into the framework of Techno-Realism.

How exactly did that come about?

In the C19th idealism was the dominant philosophy, but it was already cracking under the strain produced by the pressure of the second scientific revolution. Idealists argued among themselves as to the particular conception of the spiritual world, and there was dissatisfaction with the ascendency of Hegelianism. C19th philosophy of science, such as that of Helmholz and Mach, tended to have an instrumentalist[15] strain, but the popular movement shifted the balance towards realism and materialism; this was at first largely the work of popularists such as Auguste Comte, L. Büchner, K. Vogt, J. Tyndell, T.H. Huxley, W.K. Clifford and Herbert Spencer. The objectivity of religion was attacked by Feuerback, the Christian ethic was assaulted by Nietzsche. Materialism was assumed by

Marx, Engles and Lenin as foundational in their philosophy of material determinism. Then the academic philosophers came into the field. G. E. Moore undertook to refute idealism, and Bertrand Russell, inspired by Giuseppe Peano, attempted to create a logic founded on purely material relations, and build mathematics upon that foundation. A school of "new realists" sprung up in America – E.B. Holt, W.T. Marvin, W.P. Montague, R.B. Perry, E.G. Spauling and George Santayana. During the period between the wars Gods and Giants coexisted and fought each other within the major universities. The logical positivist movement was a decisive political force – condemning "metaphysics" – by which was meant revealed theology, Platonism and idealism – men like Schlick, Carnap, Neurath, Ayer and Wittgenstein progressively brought Western philosophy in alignment with empiricism and obliged speculative thinkers to more and more curtail their methods to those of analytical philosophy – Cartesianism was out – ridiculed as the product of the bewitchment of intelligence by grammar – and the positivism of men like Karl Popper successfully persuaded the up and coming generation that science was founded on grounds that were at once realist and empiricist. Analytical philosophy succeeded as the method of the day – all things begin with the analysis of language – logic, by which is meant, formal logic, became the core discipline. The movement gathered to a head around a single project – the ultimate expression of the aspirations of the Giants – to produce a machine that could think. Since obviously there is no way of verifying the introspective states of a machine, and Wittgenstein had "demonstrated" that there was no meaning to such states anyway, a criterion of success was needed – this was provided by the Turing test – a machine thinks, has subjective states of consciousness, if it can do everything that a human mind can do – and that means – can fool another person into thinking it is a person. An army of philosophers sprung up to justify this approach, and to argue away, on grounds of functionalism, any lingering doubts as to the validity of this claim.

The C20th also produced a significant "minority" report, especially as in the early days the advance of this positivism over idealism was fought tooth and nail. Prominent in this movement are Edmund Husserl, Heidegger, the existential movement in general (though it sometimes has its feet in both camps), Henri Poincare, Benedetto Croce and Carl Gustav Jung. But really, *looking at our planet from a distant star* can we really deny the overwhelming dominance of a cultural movement that might deviate from strict materialism in detail, but in broad outlines expresses the aspirations of the Giants – mechanism, positivism, realism, empiricism – the prioricity of logical analysis over introspection – the rejection of phenomenology, the relegation of inner states of consciousness to functions? Furthermore, all the developments outside the philosophy faculty seem to swing in favour of this movement. Computing in particular and the massive economic resources that the artificial intelligence project attracts – from driverless cars – to virtual reality – to the thinking machine itself – cognitive psychology – driven by a desire to display the mechanism of the mind – neurology exposing the putative underlying physiological basis of that claim – in theoretical physics – the quest for the various holy grails – unified field theory – quantum gravity – all potentially modelled on the instrumentalism of the Copenhagen school – but in fact culturally joined at the hip to a belief in realism. And so on. There's no escaping it.

The Giants had been in overwhelming control of the Universities long before Hamming gave his speech in 1998, so it is ironic that in *Mathematics on a Distant Planet* Hamming portrays himself as the member of a minority in contrast to "most mathematicians" who willfully persist in Platonic idealism even though they "generally realize that the Platonic view is not defensible logically". On the other hand – if there are lingering Platonists out there, then there is good reason to suspect that they haunt the mathematics faculties more than anywhere else.

The reason for this is simple – the activity of most mathematicians would commit them to a form of Platonism, if they were philosophically aware of the issues. Most mathematicians are philosophically naive in a different sense to the naivety of Hamming – but Hamming is right that real numbers and the theory of measure are *not consistent* with Techno-Realism. So if Techno-Realism is to survive, the Gods must not be allowed to resuscitate their beliefs, and implicit mathematical Platonism in the domain of real analysis must be strangled in its cradle. Hamming is speaking to young potential researchers, and an economic threat is present: "I also want to suggest strongly that <u>if in the future you want government grants and support</u>, then the latter kind of mathematics ["useful mathematics"] should get a good deal of your attention..." (My underlining.) He has already made it clear that he would bin any research that does not conform to his philosophical bias.

It is equally ironic thatHamming is scathing of formal logic: "The logician's activities seem to be irrelevant to mathematics as I understand it, but rather logic is an ideal game played by pure mathematicians for their own amusement." This attack is inconsistent, for Hamming makes liberal use of the results of mathematical logic when he suits his purpose: "The Lowenheim-Skolem paradox does seem to undercut the non countability argument Cantor gave." The Lowenheim-Skolem theorem is the corollary of the completeness proof for first-order logic, and hence not a suitable result for citation by one who rejects logic as an "ideal game". But this rejection is ironic, for it is the work of the first-order logicians that has given particular impetus to the cause of the Giants and their advance guard of Techno-Realists.

The "mathematics is first-order set theory" movement is incomprehensibly vast in its influence and intimately connected with the computing and artificial intelligence project that Hamming stands for. The core of the belief is the conviction that "Set theory is the foundation of mathematics,"[16] and "Most logicians (though perhaps not most mathematicians) are convinced that all correct proofs in mathematics could, with enough effort, be translated into formal proofs of first-order logic."[17] This movement is avowedly anti-Platonist, and espouses the formalism represented by Curry: "... we start with a list of elementary propositions, called *axioms*, which are true by definition, and then give *rules of procedure* by means of which further elementary theorems are derived."[18] It adopts the view that numbers do not exist as separable entities, but rather are the product of "juggling finite sequences of symbols"[19] or as Hamming puts it in his talk, "I am sticking with the attitude that a number is a process".

From the earliest days the "mathematics is first-order set theory" movement was associated with the analytical philosophy movement, and the assault on "metaphysics", by which is meant, revealed theology and all that sails in it. The connection between Carnap and Tarski illustrates this. "Carnap'sLogical Syntax of Language* is an example of Tarski's method in practice. Even the most highly formalized logic books had always contained passages of exposition in ordinary language ... Only if these passages can themselves be formalized, Carnap argues, will logic be wholly exact."[20]

In the period prior to 1900 logic was treated as an extension of the idealist program. That is, logic was an investigation of meanings. The syllogism of Aristotle was treated as a calculus of intensions (that is, of meanings or properties). Logic was a theory of judgement. F. H. Bradley in his *Principles of Logic* (1883) argued that true logic should not be a mere piece of formalism, it must be 'philosophical'. F.C.S. Schiller "waged a vigorous campaign against the very possibility of a formal logic".[21] Husserl called for a transcendental logic: "What the modern sciences lack is the true logic, which includes all the problems and disciplines of "theory of science" ... ; a logic that, as transcendental logic, lights the way for the sciences with the light of deepest self-cognition and makes them understandable in all their doings."[22]Poincaré mounted a scathing attack on the circularity in the definition of number offered by Peano and Russell.[23]

The logic of intensions and judgement was binned; the protests of Poincaré ignored. Dialectical argument did not prevail.

We must understand the overwhelming appeal and power of the formal logic program. It is not just a philosophical speculation but a research project bursting to the seams with practical outcomes – the goal of computing – the growth industry of the post-war period. Therefore, it was culturally destined to prevail over the philosophical objections coming from a group that was losing the game. The other point is that once the movement gathered momentum, it became self-perpetuating. This is through the institution of the research doctorate – the great families of logicians spawned by the early pioneers, themselves all associates of the Vienna Circle.

Thus, Whitehead begat Russell and Quine. Russell begat Wittgenstein; Quine, who was very prolific, begat (among others) Berry, Davidson, Craig, Kripke, Lewis, Myhill, Parsons and Wang. Craig was also the progeny of Putnam, who begat Benacerraf and Boolos. Putnam was the progeny of Riechenbach, who also begat Salmon. In a separate family, Tarski begat Feferman, Mcnulty, Monk and Montague. Feferman begat Barwise. Church was a great father. He begat Henkin, Easton, Kleene, Rogers, Rosser, Shapiro, Smullyan, Turing and Scott. Scott begat Copeland and Davies. Rogers begat Myers[24].

All of these great names areassociated with the "mathematics is first-order logic" project. On the other side, the potential critics were curiously unprolific. Frege (whose sense and reference paper makes him a possible critic) has no listed research students; neither did Gödel, that last believer in Platonism. So the Giants out-spawned the Gods. That principle applies to the map of the Universities too. As one traces the families, one sees the growth of the movement from its few initial centres such as Cambridge and Harvard, capturing the research faculties of other universities, like a virus. Viral marketing at work.

This is the socio-historical context of Hamming's paper. Far from being in a minority, he represents the main stream of philosophical thinking of the C20th, with its techno-realist bias.

So the question remains: could anyone reasonably still adhere to Plato's system, at least, in part? Has Platonism become a redundant philosophy of mathematics?

But before we tackle this question, let us deal with another issue raised by Hamming's paper, and incidentally, also by Wittgenstein's philosophy of mathematics, to which Hamming's is closely related. This is the question: is mathematics discovered or created?

This is a spurious issue, because on consideration there is only one answer: that mathematics is discovered[25]. The question is raised because the theory that mathematics is discovered is historically closely associated with Platonism. The contradiction is clear from Hamming's own paper. He states that mathematics is useful; right at the start of his talk he narrates an anecdote of how he ran a calculation to determine whether "the test bomb will ignite the whole atmosphere", concluding that "we risked all the life we knew of in the known universe on some mathematics". He not only does *not* think that mathematics is arbitrary, but he vociferously argues that aliens "facing the same laws of the physical world, their mathematics must have a good deal of similarity to ours."

So in the view of both Gods and Giants there is a reality that makes mathematics true. Mathematics is not created because it is not arbitrary – it is known to apply to the "real world". The issue, then, is not whether mathematics is discovered, but how it is discovered, and what is the nature of that reality that makes the useful mathematics true. In the case of the Gods – the Platonists – there is a supra-sensible reality of Forms that mathematics describes; mathematicians have a special faculty of understanding and/or perception – mathematical intuition – and if pressed the Platonist might claim that the visible world is made in the likeness of the realm of Forms, by a demi-urge (a bolt-on added optional extra of the Plato-package). In the case of the Giants – the materialists or quasi-materialists – it is the real physical world that makes mathematical theorems true – so mathematics is an empirical science, and proceeds through making models of portions of physical reality; it was by using one such portion that Hamming was able to predict that the atmosphere would not ignite with the test bomb.

Creativity is involved in mathematics partly because not all the systems investigated by mathematicians are applied; however, here too, the question is more imaginary than real. For whether one is a Platonist or not, once one has laid down the axioms and postulates, the theorems are discovered. Not all axioms are arbitrary, for some apply. But how do we arrive at those axioms that are not arbitrary – is it by mathematical intuition (or vision), or by some model of science?

Another thing that we can brush aside in Hamming's paper is its ugly and narrow utilitarianism – his rejection of pure mathematics in favour of "useful mathematics". The counter-argument is clear: (a) how do we know what is useful in advance of it being given? ("Useful" is an undefined term.) (b) As he admits: "The argument that in the past, pure, as opposed to directed, research has led to much useful mathematics is true" – so he refutes himself. The claim is founded upon the questionable ethic that only that which is "useful" is worth having; it denies the existence of all other values, and that, surely, must be open to question, even from with the camp of the Giants.

Returning, then, to our main question: could anyone reasonably still adhere to Plato's system, at least, in part? Has Platonism become a redundant philosophy of mathematics?

This question splits into two points: (1) Has the Platonic doctrine of universals been so decisively refuted that no rational person could believe in it? (2) Is mathematics first-order set theory? Wrapped up with this second question is the question: what is the ontological commitment of first-order set theory, and if mathematics is not first-order set theory, does that commit mathematics to some form of Platonism?

Dealing with the first question – does nominalism reign supreme? My impression is that in the minds of majority of Giants nominalism does reign supreme, but this is not because of the cogency of any argument provided against universals (or abstract entities, if we need to distinguish them from universals), but simply because *it is not considered*. I think it is an assumption, for instance, of the formalist school of the philosophy of Mathematics[26] that saying "numbers are processes" is to state a manifest truth without epistemological payload. For example, Benacerraf writes, "To *be* the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth. And to *be* the number 4 is no more and no less than to be preceded by 3, 2, 1, and possibly 0, and to be followed by ... *Any* object can *play the role of 3*."[27] In this case Benacerraf simply thinks that asserting that numbers are processes circumvents the need to explain what is universal about such processes, and how we *know* that "any object can play the role of 3". The contradiction is shown by his remark: "numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure*..."[28] So what is that abstract structure other than a universal?

So this is "Ostrich nominalism" – just bury one's head in the sand and forget it. Numbers are processes; numbers do not exist; no explanation needed; end of story.

A glance at the Stanford article on nominalism[29] suggests that the issue of nominalism is a lively debate, and hence, by implication, that Platonism is still a tenable theory. That is right to an extent, but possibly underestimates the effect of Ostrich nominalism; that a small sample of professional philosophers undertake to build a cogent nominalism in reply to Platonic argument is not a gauge of the state of belief among the majority of the Giant movement.

But one of the arguments cited against universals in the Stanford article deserves special attention:

> Another common and widely discussed argument against abstract objects is an epistemological argument. The argument is grounded in the thought that given that abstract objects are causally inert, it is difficult to understand how we can have knowledge or reliable belief about them. Sometimes a similar argument is advanced according to which the problem with Platonism is that, given the causal inertness of abstract objects, it cannot explain how linguistic or mental reference to abstract objects is possible (see Benacerraf 1973 and Field 1989, 25–7).[30]

This line of thinking is disingenuous, begs the question, and illustrates the cultural domination of the materialist way of thinking. For it is precisely the theory of Plato that the ability to *know* universals (and/or abstract objects) is *proof*, via the transcendental deduction, of the non-materiality of the soul. Therefore, it is not an objection to the Platonic theory that “it is difficult to understand how we can have knowledge or reliable belief about them.” Plato says, not everything that exists is a causal object; Benacerraf and Field say, everything that exists is a casual object. But it is Benacerraf and Field that have the problem – of universals. Hence the term “problem”, which reflects the bias of the C20th.*It is not a problem for Plato*.

The author of the Stanford article also reflects this bias, for he immediately writes, “Admittedly these arguments do not conclusively establish Nominalism but, if they work, they show an explanatory *lacuna* in Platonism. The challenge for the Platonist is to explain how knowledge of and reference to abstract objects is possible.”[31] It is not a challenge at all for the Platonist, for *that is Plato’s theory*, his whole point. And, amusingly, with the exception of “Platonism”, there is not a single reference to Plato in the whole article, even when the problem of participation is adumbrated.

Rather than pursue the current nominalist theories in vogue (Armstrong, Goodman), let me ask the question – did Wittgenstein conclusively nail nominalism?

> §66. Consider for example the proceedings that we call "games"...[to] look and see whether there is anything common to all. ... And the result of this examination is: we see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities.
> §67. I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way. – And I shall say: "games" form a family.[32]

This is not convincing to a Platonist. A universal is a structural similarity between two particulars, it is a real similarity between them that does not imply that those two particulars are identical in every way. A universal is part of our understanding of the world, and it is that understanding that needs explanation. However, Wittgenstein’s theory is more internally consistent than this objection might suggest, for via his doctrine of “meaning is use” and his arguments against “private language” he argues that conscious, subjective states of understanding are irrelevant to communication. Nonetheless, the Platonist would clearly not agree.

For it is a question of what is the ground on which one stands. If, for instance, one takes the "real" external world as given, as an unambiguous object of common sense knowledge, then it is difficult to account for knowledge of other people's subjective states, or to explain how one knows one's own independently of a language based on correspondence and/or coherence with that given real physical world; hence meaning is use. But one is *not obliged to stand upon this ground*. A phenomenologist will take the unity of subjective consciousness as primitive, and *starting from there*, with meaning given, he then explores outwardly his philosophical world. It is possible to interpret everything that Wittgenstein ever wrote as an attack on Plato; and yet, precisely because of that, the common ground between them is a best marginal, and *Plato is not so much as scratched by the whole endeavour*. Wittgenstein is speaking only to those who are already converted.

So that deals with the first question. To conclude: on examination, and despite the appearance that universals have been cast aside, no overwhelming cogent argument against universals does currently exist. It is possible to be a Platonist in this sense. Hence, Platonism in mathematics *is not dead*. With Benacceraf's statement that "you merely characterize an *abstract structure*" Platonism gets back off the ground. Conclusion: not proven.

Now we turn to the second question: is mathematics first-order set theory? The examination of this issue turns out to be even more profound.

For firstly, we must examine the question of the ontological payload of set theory and of first-order logic.

Now is first order logic a calculus of propositions, individuals and properties, all of which qualify as universals? This is already tricky, but no more of a problem that that of universals themselves; if there is a nominalism with regard to universals in general, that will apply here. Formalists replace propositions, individuals and properties with syntactic objects of mere formal manipulation: sentences, terms and predicates. So in this respect there is a problem, but not a new one.

When we come to sets, there is a new issue. For a set, being a collection based on the primitive notion of set-membership, prima facie requires firstly some kind of basis in intuition for how we understand that primitive, and secondly, being a non-material collection, is a good candidate for an abstract entity as much as a universal. Indeed, if we can argue that the identity of the set $\{a,b\}$ to the set $\{b,a\}$ does not involve the relation of universal to instance, we can see sets as individual abstract entities rather than universals per se. So the Platonist has good grounds here for support; as Parsons remarks, "set theory is the very paradigm of a *platonist*theory"[33]. Prima facie, set theory requires a Platonist epistemology, *and the transcendental deduction goes through*.

The specific butt of Hamming's attack is the author of set theory, Georg Cantor, whom Hamming would happily have seen confined to an asylum. Since set theory is generally extolled by the Giants, it is an irony that author of set theorywas a member of the party of the Gods. For example, his biographer, Joseph Dauben, explains how correspondence between Cantor and Hermite shows that both were Platonists and that Cantor decidedly linked this mathematics to his belief in God and reading of St. Augustine[34], who is famous for his fusion of neo-Platonic thought with Scripture. Cantor's concept of the actual infinite is clearly another candidate for a specific abstract object, and one not accessible to Man ifMan is only conceived of as being the mechanical product of material forces.

The strain is so palpable that the Techno-Realist wishes above all to dispense with set theory altogether, and this tendency is well-evidenced by Hamming's speech. One step in this direction is to recast mathematical theories in first-order logic alone, without the paraphernalia of sets.[35]Another approach is to treat the symbols of set theory as *without reference* – that is, as counters in a logical game, and maintain a syntactic approach. For the Platonist, Cantor's symbol for the actual infinite, ω, stands for an object that we grasp through *conception*; for a formalist it is simply a physical counter that is manipulated according to rules that could, in principle, be programmed into a digital machine. However, this approach must be accompanied by a good deal of Ostrich nominalism, for the universaling tendency even of such counters has already been noted.

There is a good deal of Ostrich nominalism at work with regard to set theory. That a standard text book of set theory should fail to discuss the issues of the ontology and epistemology of sets can be attributed to the focus on the results that arise from the set membership relation. On the other hand a title such as *Set Theory and its Philosophy*[36]suggests that the reader is invited to feast on a discussion of Platonism and universals; the discussion is confined to the Preface, and is cursory. The author does reject formalism, but he does not provide an epistemology for sets.So where is the philosophy?

All contemporary academic philosophers (perhaps with private doubts) are affiliated to the school of the Giants. But the philosophical situation is much more problematic than the popular version of the philosophy of the Giants would allow. All the time, on consideration, the fundamental basis of the philosophy of the Giants – that only the physical has real existence – comes under strain. One battle ground is the "problem" of universals. Platonism with regard to universals and/or abstract objects has a tendency to rear its ugly head at every twist and turn, and with it comes *the transcendental deduction*. (Not a bolt-on extra feature.) The whole aspiration of the Giant movement, to build a machine that thinks, to create life from silicon, is bust apart in this single movement – for how can you build the non-material out of the material? Mathematical intuition is the very rock on which the aspirations of the Giants flounder. Nonetheless, a discussion of sorts is conducted, for it is inevitable. At those moments one gets a cursory exposition, with phrases like, "A Platonist would say ..." and "A formalist would say ...". Then the author, without making a definite statement of his own, concludes by acknowledging something like, "it is a problem" (for whom?) and then moves on to some less controversial topic, such as whether sets can be coherently said to constructed in stages. Every now and then, one of these Giants betrays a strong Platonising tendency; then, depending on the nature of the statement, one or other of the Giants pops up with a rebuke. Thus, Wang (possible neo-Platonist[37]): "The overviewing of an infinite range of objects presupposes an infinite intuition which is an idealization."[38] Response, Parsons: "An example is the subtle and interesting treatment of Professor Wang. This conception is more metaphysical, and in particular more idealistic, than I would expect most set theorists to be comfortable with."[39]

There is one philosopher – undoubtedly a member of the Giant party – who did give considered attention to the problem of universals posed by set theory: that is, Quine. His answer is clever and subtle. Though constructed as a defence of positivism, it does more than any other "solution" to the "problem" to expose the philosophical issues, and will lead us to even greater revelations. But first to an exposition of his solution.

The first thing he does is to offer a criterion of ontological commitment. This is the famous, "To be is to be the value of a variable". This says that if you use an expression such as, "There exists an *X*..." or "All *X* are *Y*", then you are ontologically committed to saying that the kind of entity that *X* names *exists*. On that ground, since first-order set theory does contain statements of the form, "There exists a set that is the union of the members of two other sets" and the like, then set theory is ontologically committed to the existence of abstract entities that individual sets name.

This is a devastating admission from a man who is a champion of the cause of the Giants (a giant among Giants). It is also not Ostrich nominalism. And yet, Quine still remains a nominalist. How does he achieve this?

He embeds his theory of sets into a theory of science as a whole that allows him, as he at least thinks, to claim that all such abstract entities are convenient fictions – part of a myth.

> Now what of classes or attributes of physical objects[40], in turn? A platonistic ontology of this sort is, from the point of view of a strictly physicalistic conceptual scheme, as much a myth as that physicalistic conceptual scheme is for phenomenalism. This higher myth is a good and useful one, in turn, in so far as it simplifies our account of physics. Since mathematics is an integral part of this higher myth, the utility of this myth for physical science is evident enough. In speaking of it nevertheless as a myth, I echo that philosophy of mathematics to which I alluded earlier under the name of formalism."[41]

This would appear to be a form of "having one's cake and eating it", so Quine needs to further explicate how this can work. In his essay *Reification of Universals* he explains why this myth does not force an ontological commitment. Referring to a statement letter *p*, he writes "even the quantifiers '(p)' and '$(\exists p)$'[42] happen to be reconcilable with nominalism if we are working in an extensional system". This device is attributed to Tarski. Another device presented in the same essay is to introduce the stage theory of sets; this gives "some semblance of control" and is based on the principle that "classes [sets] are conceptual in nature and created by man"[43].

A Platonist could shrug his shoulders to all of this. Basically, on an unbiased reading of the situation, the dialectical balance between the Gods and Giants is much more even than it appears to be. As already mentioned, the cultural reaction to the cosmological revolution brought on by the second scientific revolution of the early C20th gave a tremendous boost to the cause of the Giants. However, the core of the Platonic defence of the cause of the Gods was *never really touched* by it, and it was a cultural-historical accident that no "champion" capable of putting this point forward arose at the time. There was no St. Augustine. I have mentioned Husserl, Heidegger, Croce and Poincaré, but none of these were able to reach down to the masses to make them more aware of the residual argument in defence of the "divinity of human nature". The person who got closer in this respect was Carl Gustave Jung, who did found a popular movement. But despite an early self-training in philosophy, he sometimes shied away from direct confrontation with the spirit of the age, and sometimes sought rapprochement with it. He never had anything to say about universals. Nonetheless, he is the most important of those who presented the minority report, and I shall return to his significance later.

Thus, for a Platonist, Quine's attempt to evade the Platonic obligations of first-order set theory are unconvincing. But be that as it may, for a nominalist, it may work. On the other hand, what Quine says about second-order logic really puts the cat among the pigeons and goes to the heart of Hamming's hysterical reaction to such things as real numbers, set theory, logic and the Lebesgue measure.

Quite reasonably and "from a logical point of view" Quine identified quantification as the measure of ontological commitment. To refresh this point: if one says "there exists an X such that..." then one is committed to saying that X exists. If X happens to be a universal or abstract object (or both) then one has the epistemological problem of explaining how X can be known. Then, lurking in the background is the transcendental deduction, and you have established the essential non-materiality of the mind. So this criterion of quantification is very important.

In the first-order predicate calculus, one says, "There exists an individual *x* such that *x* has the predicate *P*." So far, for the nominalist, so good. There is no quantification over the predicate *P*, and the *x* may be taken to refer to individuals, that is, specific material objects in space and time that are causally effective.

In the first-order set theory, one says, "There exists a set α such that α and the predicate *P*." This is trickier. The set is an abstract object, and here one has to introduce the myth that Quine describes. It may work, and then again it may not. This is dialectic.

In second-order predicate calculus, one says, "For all predicates *P*, such and such...". Now one quantifies over a predicate and there is no evading the ontological payload: it *must* have a reference, and that reference is a *property*; it is both a universal and an abstract object, and you are buried alive in Platonism. This is a body blow to the cause of the Giants. Let us have this in Quine's own words:

> Now the maneuver of extending quantification to predicate letters, as a means of expanding quantification theory into class theory, can be represented as a provision merely to allow predicate letters all the privileges of the variables. ...
>
> Such extension of quantification theory, simply by granting the predicate variables all privileges of '*x*', '*y*', etc., would seem a very natural way of proclaiming a realm of universals mirroring the predicates or conditions that can be written in the language. ...
>
> But in any case universals are irreducibly presupposed. The universals posited by binding the predicate letters have never been explained away in terms of any mere convention of notational abbreviation, such as we were able to appeal to in earlier less sweeping instances of abstraction.[44]

No clearer exposition of the point has ever been given – I am immensely grateful for it. So the conclusion is: *the Giants must avoid second-order logic at all costs whatsoever*. But can this be done?

Answer: not if you are a mathematician and do real analysis. Some mathematics can survive in first-order logic (or first-order set theory), but *not all of it*.

To explicate this point further: when Wolf states that all mathematics is first-order set theory[45] he is plainly wrong. It is manifestly a false statement, and not even remotely defensible upon reflection.

The language of mathematics in general is a combination of second-order statements within the context of natural language arguments. Many text books are about structures definable in first-order set theory, as indeed, text books about set theory are (!), but the gloss and the main text even in those books is not written in that language. This could be regarded as a convenient presentational device, but that is not the case in general.

Actually, the situation is more or less as follows.Mathematical discourse is about two kinds of "object".

> (1) There are those objects that are in some sense finite. Sometimes such objects have symbols that on a Platonist interpretation would refer to infinite collections or some such, but on examination they can be given a finitary interpretation. This will depend technically on whether such theories (when formulated in first order logic) are complete (stronger property) or compact (weaker property). The mathematics involved is very closely allied to combinatorics. The structures and inferences about such structures can be described using first-order logic and/or first-order set theory. They are also candidates for algorithms.

(2) Then there are those objects that are not finite and cannot be reduced to a finite object. These divide into two: those objects that are irreducibly potentially infinite, and those that are irreducibly actually infinite. Such objects can only be approached by non-mechanical principles of reasoning.This point ought to be transparent; however, the essentially non-mechanical nature of the reasoning is obscured by the nature of mathematical rigour – the standards of proof. These standards require that all theorems be proved by detailed steps from clearly stated postulates, definitions and axioms; the steps are minute inferences conducted within some kind of algebra. So the very nature of mathematical argument obscures the nature of the inferences being conducted.

As the above depends on the distinction between the potential and actual infinite – let me explain that as well. In mathematical discourse we meet two differing conceptions of the infinite:the potential infinite and the actual. The potential infinite is illustrated by the inexhaustibility of counting; for no matter how large a number we have reached it is always possible to count to a higher one by adding one more. In the actual infinite we conceptualise the entire process of counting as a *completed totality*.

Now to clarify these points further. In mathematics, in addition to first-order reasoning, we see the following forms of reasoning.

(1) In number theory and its applications to real analysis inferences based on the axiom of complete induction:

For all properties *P*; if *P* holds of 0, and if, given *P* holds of an arbitrary number *n* then*P* holds of the successive number $n+1$, then *P* holds of all numbers.

This axiom is irreducibly second-order as it quantifies over properties of numbers, which are consequently universals. The domain (scope of the principle) is the collection of all natural numbers, which is a potentially infinite collection.

(2) In analysis inferences are derived from the axiom of completeness.

To explain this, I need first to remark that if in set theory one quantifies over sets of sets, then that is equivalent to a second order statement. Real numbers are conceived as the limits of the sequences that converge upon them. For example, Leibniz's formula, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots=\frac{\pi}{4}$, is a sequence that may be used to compute π. There are several equivalent formulations of the completeness axiom; one is the Cantor nested interval principle

Given any nested sequence of closed intervals in $\mathbb{R}$,
$[a_1,b_1]\supseteq[a_2,b_2]\supseteq\ldots\supseteq[a_n,b_n]\supseteq\ldots$,
there is at least one real number contained in all these intervals: -
$\bigcap_{n=1}^{\infty}[a_n,b_n]\neq\varnothing$

This is irreducibly second-order because it quantifies over "any nested sequence of closed intervals". This formulation displays clearly that the second-order axiom of completeness brings into existence real numbers. There is no such device within first-order set-theory. So first-order set theory actually has no real numbers; second-order theory with the axiom of completeness does have them. A clear and unbridgeable distinction. The second order

> theory thus generates models the arithmetical continuum, and gives an account of continuity. There is no such model in first-order set theory.

I also believe that there are other identifiable patterns of interference in mathematics that cannot be reduced to a first-order statement and/or inferences.[46] However, as our theme is the unavoidable presence of Platonism in mathematics, I shall not digress further on this topic here.

Regarding number theory and the second-order axiom of complete induction, set theorists claim that they can provide an axiom schema for induction which is *not* second-order; also they claim to derive a principle of "transfinite induction" from within set theory. To pursue these topics would be too much of a distraction in this context, for the main thrust of the exposition relies on the axiom of completeness and the real numbers that it defines.

For the uninitiated, calculus, that is, analysis, is founded upon the axiom of completeness. It is this axiom that constructs the continuum,tells us what a real number is; a real number is what gives limits to sequences; and without it, *there is no calculus*.

We have reached the point of discussion of real numbers. This is what Hamming would have us believe about real numbers:

> Thus a *computable-number* is a number for which there is some program to compute it on some Turing machine to as many (necessarily finite) digits as you specify. Thus Turing, in trying to get (ideal to be sure) mechanical proofs, changed the concept of number from its *representation* to the corresponding *process* of getting as many digits as you please. The number π is now a program, say the program that generated some 6 billion digits of it, and is no longer the original infinite representation. Thus Turing introduced another modest step away from the actual infinite and back to the finite, but unbounded: the potential infinite of Aristotle as opposed to the actual infinite of Cantor.

Consider two distinct sequences that converge on the "same irrational number", say $\sqrt{2}$. But if we do not have a notion of a two sequences being actually infinite, then it is strictly not possible to talk of them as converging to the same limit, for $\sqrt{2}$ does not exist. That also means that the equation $x^2 = 2$ has no solution. An alternative way of putting this is:*there are as many* $\sqrt{2}$ *as there are sequences that approximate to it*. In order to prevent this situation we must introduce a <u>unique number</u> that is the identical limit of these differing sequences. This transforms the intensional (that is conceptual) notion of sequence into the extensional notion of an equivalence class of ordered sets and into an actual infinity. Cauchy added $\sqrt{2}$ not as the name of a sequence but as the name of the limit of a sequence. Under this conception, different sequences can have the same limit. The completeness axiom asserts the independent existence of limits, independently of the sequences that approximate to them, so explicitly allows for quantification over limits as now real numbers. From this arises the concept of the arithmetical continuum.[47]

So if we following Hamming's theory $x^2 = 2$ does not have a solution. That is not exactly a satisfactory position, and is clearly an arbitrary imposition on mathematical physics.

Relating all this to Hamming's specific comments quoted above – If π is a program, then there are as many π as there are programs that compute π. We can never say that two such distinct programs compute the same π. Turing did not change "the concept of number" from its limit to corresponding process. He made no statements whatsoever with regard to this issue. The theoretical framework in which some numbers computable and some are not is part of our understanding of continuity. We cannot dispense with it, because in that case *our analysis becomes* *meaningless*. How can there be an analysis without the notion of two sequences converging on the same number? All of mathematical practice decries against it.

The Axiom of Completeness is second order and quantifies over properties of individuals. To quantify over something is to begin to accept that thing as "real" – as part of the world. But properties seem to belong to the intellectual world of subjective interpretation rather than to the "real world" of things. In the late C19th idealism was the dominant philosophy, and the interweaving of *ideas* with *things* would not have appeared strange or problematic, had the matter been considered. But in the early C20th positivism overwhelmed idealism, and the idea of intellectual concepts being integral of science came to be seen as problematic. Second order properties were dropped and the spirit of the age demanded that science be conducted in first-order logic.

Mathematics still proceeds according to both traditions. Hence, Hamming on one side of the fence, his colleagues, still working in the C19th tradition, on the other.

This relates to different models of how science works. Implicit in the C19th approach is the view that science arises from the interaction of the questing mind with the phenomena. The central concept is observation. In the C20th, with its positivism, the consciousness part of this equation was eliminated. Then the problem is one of how to describe the "real world" in a language that does not bring consciousness back in. This entails that universals must be eliminated. Whatever the residual problems associated with first order logic are, the prime motivation, philosophically, for using first-order language is, as per the arguments of Quine, it at least has some claim to be consistent with the triad of nominalism, formalism and empiricism. If we can build a model of reality out of first-order language we might, just might, be able to claim that it is neutral as to the claims of phenomenology. Then we can possibly construct consciousness out of the functions defined in that language. Frankly, for me, it just ain't good enough, but then, I am not a nominalist.

Let's deal with the comment of Hamming on the Lebesgue integral: "Indeed, for more than 40 years I have claimed that if whether an airplane would fly or not depended on whether some function that arose in its design was Lebesgue but not Riemann integrable, then I would not fly in it." Firstly, the mention of Lebesgue in this is spurious. The equation, Riemann = good, Lebesgue = bad has *nothing to do with it*. For the Riemann integral is every bit as much founded on the Axiom of Completeness as is the Lebesgue integral. The intellectual movement that sees one substitute the Lebesgue integral for the Riemann integral is perhaps driven by theoretical concerns, but its practical uses can also not be discounted. These are explained in Bressoud's excellent book[48]. I will just summarise one of these.

Term-by-term integration is the interchange of summation and integral signs.

$$\int\left(\sum f(x)\right)dx = \sum\left(\int f(x)\,dx\right)$$

This is used in the construction of the Fourier series. Conditions for when term-by-term integration can be given for the Riemann Integral, but these conditions are unduly "complicated". The interchange of the summation and integral signs is of immense practical importance; Fourier series are fundamental to physics in all respects. Hence, there is no utilitarian reason for rejecting the investigation of the Lebesgue integral if it simplifies term-by-term integration. Hamming is misinformed.

The driving force behind Hamming's speech is philosophical and not utilitarian. He is a member of the party of the Giants, and he is sure that all that exists is material. He is quite right that the

Lebesgue integral is wedded to Platonism; he imagines that the Riemann integral is not. It is a fantastic piece of polemical writing, a sermon written from the heart, but not from knowledge.

We now reach the point where we can deal with one residual issue raised by Hamming's paper. This is the ostensible motivation behind the paper – its wrapper. This is the question – what kind of communication would we expect with aliens from a distant planet? His conclusion: "facing the same laws of the physical world, their mathematics must have a good deal of similarity to ours."

Strictly, this is really a marginal problem, since the questions raised by his paper concern the philosophy of mathematics on our planet. But let us digress.

The question raises the problem of what universals actually are. Plato argues that they belong to a supra-sensible realm of Forms, which are abstract entities, and accessible to human minds through some intellectual faculty. Thus, Plato would probably agree with Hamming that the extra-terrestrials would have the same mathematics as we do, since they experience the same Forms.

However, if I apprehend an object, say with my sight, there is some claim to that object being dependent on the mind – it is at least *before* the mind if not *in* it.[49] Hence, if there are universals, these could be mind-dependent "categories" of understanding, and this is, of course, the theory of Kant. Then the Kantian theory essentially makes our categories dependent on our psychology. That applies also to our understanding of the continuum, and to the real analysis we build out of that. Hence, in the broadest sense of possibility, *it is possible that aliens would have a different mathematics to ours*, provided their psychology was fundamentally different to ours. Now the main point I am making here is that Hamming's paper reflects a deep and frankly ugly dogmatism. He simply closes off discussion. It is an aggressive one-sidedness behind the mask of scientific rationality.

In this spirit there is also his support for the statement of Galileo: "Mathematics is the language of Science".[50] This is *not* at this time an empirical statement. At this time observation has not confirmed it. In one respect is an organising principle, one that acts as a motivating belief behind research in mathematical physics. In another respect, *it is a cry of faith*. In that case, there is no rational foundation for the belief in this statement, so it is founded on irrationality.

In a sense *I do like it*. For Hamming is a man of blood, a man of conviction and cause, and a man of faith. That is why, at the close of his life, he chooses to preach a sermon. He articulates his faith.

What I don't agree with in Hamming is his narrow-minded interpretation of rationality. What I don't like about his paper is his aggressive dogmatism. Honestly, the story about the paper that went into the bin is *shocking*. And he is shameless about it.

In summary, the C19th was the century of the Gods. These proponents of idealism failed to capitalise on their advantage – they squabbled among themselves and failed to unite around the core arguments. The C20th saw a revolution in culture that swept away the Christian cosmology and replaced it with the Big Bang and Evolution. In the wake of that revolution, the C20th saw an onslaught on C19th idealism and also swept it away.

But the revolution, at least in the philosophical domain, went too far. It exaggerated the victory of the Giants. While they took over the academic highground, the objective question – has the core of Platonic epistemology been refuted? – was not adequately addressed. The dialectic became imbalanced, but *unfairly*. There is nothing overwhelming in the contemporary statements to refute the rationality of still believing in the Platonic theory of universals, *and the transcendental deduction that it brings*

And I do believe it.

It would be an exaggeration on my part to suggest that the discussion of universals in the work of the Giants, where it is discussed, is anything other than very level headed. On the contrary, the work of Quine is particularly clear and even inspiring; I see Dummett's article on abstract objects in his work on Frege[51] as equally balanced and perspicuous. These are samples only. But this does not really get to the nub of the problem, which is cultural bias. While the opinion of Quine on the issue is not in doubt, even in the work of Dummett one reads in effect, "the Plationist would say this..." and "the formalist would say that...". So where does Dummett stand on the issue? And what happened to the transcendental deduction? It is the very phrase "the problem of universals", like "the problem of consciousness" that reveals the whole cultural bias – it is not a problem for Plato.

We know how it comes about. I'm not saying that academic philosophers have no blood in their veins, but perhaps they have less blood than other men. It is the essence of living to be committed to a cause, not to stand on a fence, or, even, to pretend one stands on a fence.

For instance, as a man I want to know about death. Now, personally, I have long given up worrying about death, but still the question does concern me *as a man*. I know that death is the great unknown, and that actually, like all men before me, I'll have to go into that plunge into the unknown, in the ultimate sense unknowing. Notwithstanding, I would like to know whether I have any grounds for believing I am a soul, and whether my personal consciousness will survive. Again, I am not so personally concerned in this question, but that is partly a product of the stage of understanding that I have reached. For I know that the self is not a material object, hence exists in eternity, that is, not in space and time. My body is phenomena only, and hence my death does not affect me in my essential being. I am timeless; what, then, has an event in time to do with my essence?[52]

It is an important issue – not just for me, but for all humanity – that *we get the philosophy right*. Ultimately, whatever we think about freewill itself, we are all practically free to *choose the ground we stand on*; hence, I cannot refute a materialist, and I have no desire to do so. Nonetheless, I do want a level playing field – by which I mean, that the validity of an argument should not be exaggerated, and that issues that are problems for everyone should not be brushed aside, Ostrich like. One such issue is – *how is it possible that my world is intelligible to me as a conscious being?* Another issue is – *what is the origin of mathematical knowledge?* Since I don't in the least expect to brush aside my philosophical opponents, I am not looking for, or expecting, the overwhelming victory of my subjective point-of-view – for ultimately, it is subjective if I say that all knowledge is based in my direct self-consciousness of the phenomena, and another person, say Wittgenstein, says it is not. What is a problem is if Wittgenstein, or some other philosopher, refuses to acknowledge his own subjectivity, and hence refuses to accord even the decency of rationality to the other side. Lock them up, if they don't agree with you. Confine them to the mad house. So I agree with Jung, "Rationalism and doctrinarism are the diseases of our time; they pretend to have all the answers."[53]

However, I do care about other people, and for that reason too I'd like the arguments to be considered on their own merits, and not just put in the bin on the basis of prejudice. I owe it to other people. Ido believe that people are happier if they believe they belong to the infinite – in other words, do not fear death.

> The decisive question for man is: Is he related to something infinite or not? That is the telling question of his life. Only if we know that the thing which truly matters is the infinite can we avoid fixing our interests upon futilities, and upon all kinds of goals which are not of real importance. Thus we demand that the world grant us recognition for qualities which we

> regard as personal possessions: our talent or our beauty. The more a man lays stress on false possessions, and the less sensitivity he has for what is essential, the less satisfying is his life. He feels limited because he has limited aims, and the result is envy and jealousy. If we understand and feel that there in this life we already have a link with the infinite, desires and attitudes change. In the final analysis, we count for something only because of the essential we embody, and if we do not embody that life is wasted. In relationships to other men, too, the crucial question is whether an element of boundlessness is expressed in the relationship.[54]

Now it is obvious that nothing really can prove or disprove to me whether there is an infinite or not. It is a matter of faith. Nonetheless, philosophy is not futile, for it can act as an articulation of faith, and lead one to be more conscious of the ground upon which one stands. Culture too, acts in this way. Thus, if my culture tells me incessantly that I am a machine, and conjures up the spectre of a thinking mechanism, if it gives me image after image of dead bodies exhibiting their mechanical parts, I may be lead falsely <u>not to consider what is infinite in myself</u>, and the result may be a cramping of my spirit, producing an ugly neurosis and false dealings with my fellow men. So it is important that my culture should not artificially exaggerate conclusions.

What could prove to me that I have a soul, that I exist in eternity, and not merely as a spatial-temporal entity? Experience alone is not enough. Suppose an angel appears to me – then I must ask myself - am I an inspired visionary or someone suffering from a mental illness? Therefore, without the consolations of philosophy I am at a loss to interpret my own experience. Something appears beautiful to me. What does that mean? I need a logic with which to comprehend my subjective experiences.

In Plato's epistemology and his consequent transcendental deduction, we find the logic that is lacking in mere phenomenological experience. Herein is an argument that shows me that *I am not merely a material thing*. With that solid rock to stand upon, I start to be able to navigate in life.

Melampus

Appendix – can limits be constructed on the basis of the potential infinite?

There is a question as to whether, regardless of what Cauchy himself thought, the system of Cauchy can be construed as a system based on concept of the potential infinite by use of the epsilon-delta notation. The test is quantification – once one quantifies existentially over a limit (number) one brings it into existence so to speak. The question is whether in the system of Cauchy it is possible to avoid altogether an existential quantification: to affirm that only the sequence exists and not its limit, and that all talk of limits is a façon-de-parler, which is, I believe, the received wisdom on the interpretation of the so-called rigorous foundation of mathematics introduced by Cauchy.

In analysis sets begin their existence (historically) as sequences that have limits. Without sequences whatsoever, there cannot be any form of irrational number. What one gets is incommensurable ratios. All one can say is that the ratio does not exist because it cannot be measured. Or possibly, the ratio does exist but nothing determinate can be known about it, because it cannot be measured. But we can measure it. This is achieved by a sequence of ratios that are successively better and better approximations. The proof that the approximations are better and better uses only the mathematics of rational numbers (ratios). So we can now add the sequences as uncompleted objects. Let us say $\sqrt{2}$ is the name of such a sequence. It does not name the limit but only the sequence. But the sequence has an essential incompleteness being indeterminate because it embraces a potential infinity. It is an inductive rule for generating ratios and is connected to the process of counting. The sequence cannot be grasped without the rule, which is the concept. Therefore $\sqrt{2}$ conceived merely as an essentially incomplete sequence is an intensional object though it does have an extension. The extension is also essentially incomplete, so we cannot identify the extension with the intension, nor can we say that the former captures the entire meaning of the latter.

It is clear that in the concept of Cauchy and others $\sqrt{2}$ is added not as the name of a sequence but as the name of the limit of a sequence. Note, different sequences can have the same limit. With the intensional concept of sequence, each sequence is different, *so there are as many* $\sqrt{2}$ *as there are sequences that approximate to it*. In order to prevent this situation whereby different sequences with the same limit are different numbers, we must introduce a unique number that is the identical limit of these differing sequences. In order to do this, we must extend the sequence to include all its members, thus treat its course of values as a determinate extension, which implicitly transforms the intensional notion of sequence into the extensional notion of ordered set. By the time of Dedekind, this step has been definitely taken. We now have sequences as existent objects and separately their limits, also as existent objects. Differing sequences may have the same limit, but not conversely. A limit is only given by a sequence. No sequence, no limit.

Conceptually a limit is a set of sets. Take all sequences having the same limit *l*. Then by forming the equivalence class (set) of all such sequences we create a new object, named by the limit *l*. So we can always divide through to obtain a single object that is a set of sets all of which have some defining property. So the theory of limits is prima facie a second-order concept. Actually, the completeness axiom, also a second-order concept, circumvents the need to introduce a set of sets. The completeness axiom asserts the existence of limits, so there is no need to divide through and use the equivalence set. It asserts more, because it asserts the independent existence of limits, independently of the sequences that approximate to them, so explicitly allows for quantification over limits as now real numbers. So Dedekind (et al.) decidedly marks the shift from the idea of limits existing individually to the idea that they exist as competed totalities – for example, the set of all real numbers.[55]

Notes and References

[1] R. W. Hamming, *Mathematics on a Distant Planet,* American Mathematical Monthly. Vol. 105. No.7
[2] Philip J. Davis: Richard Hamming: An Outspoken Techno-Realist, Society for Industrial and Applied Mathematics, November 16,1998. https://www.siam.org/news/news.php?id=893
[3] John, Chapter 1, Verse 1 – The New English Bible – Oxford University Press, Cambridge University Press 1970.
[4] Thomas Hobbes, Leviathan, Part I, Chapter 4, *Names, Proper and Common*. First published 1651.
[5] David Hume: *Enquiry Concerning Human Understanding,* II, 13.
[6] Ibid. 17
[7] Ibid.II, 16. "Suppose, therefore, a person to have enjoyed his sight for thirty years, and to have become perfectly acquainted with colours of all kinds except one particular shade of blue, for instance, which it never has been his fortune to meet with. Let all the different shades of that colour, except that single one, be placed before him, descending gradually from the deepest to the lightest; it is plain that he will perceive a blank, where that shade is wanting, and will be sensible that there is a greater distance in that place between the contiguous colours than in any other. Now I ask, whether it be possible for him, from his own imagination, to supply this deficiency, and raise up to himself the idea of that particular shade, though it had never been conveyed to him by his senses? I believe there are few but will be of opinion that he can: and this may serve as a proof that the simple ideas are not always, in every instance, derived from the correspondent impressions; though this instance is so singular, that it is scarcely worthour observing, and does not merit that for it alone we should alter our general maxim."
[8] Plato, Theaetetus, 151E, from F. M. Cornford, *Plato's Theory of Knowledge*, Routledge & Kegan Paul, London and Henley, 1979 p. 30 et seq. (First published 1960.)
[9] I have taken the term from Kant's *Critique of Pure Reason*. Kant advances a related but different argument in the *Critique*, one in which he deduces the transcendental independence of the self as an object not located in space and time from the subjective phenomenological experience of successive moments in time. The general pattern of the argument, common to both Plato and Kant, is to derive a metaphysical conclusion about the nature of the human mind from what its faculties are.
[10] Plato, Theaetetus, 151E, from F. M. Cornford, *Plato's Theory of Knowledge*, Routledge & Kegan Paul, London and Henley, 1979 p. 185. (First published 1960.)
[11]It is not a new idea. Karl Vogt wrote in *Physiological Epistles* (1847) that 'the brain secretes thought, just as the liver secretes bile'.
[12] Cornford subtitles the second of the *Sophist* from which this quotation is extracted, "*The Battle of Gods and Giants: Idealists and Materialists*" p. 230. from F. M. Cornford, *Plato's Theory of Knowledge*, Routledge & Kegan Paul, London and Henley, 1979 p. 228.
[13] Plato, Sophist, 151E, from F. M. Cornford, *Plato's Theory of Knowledge*, Routledge & Kegan Paul, London and Henley, 1979 p. 230. (First published 1960.)
[14]David Malet Armstrong, celebrated materialist, founder of the Australian school of materialism, author of *A Materialist Theory of the Mind*.
[15] The doctrine that scientific theory is an instrument for explaining phenomena, compatible with idealism, and opposed to scientific realism.
[16] Kenneth Kunen, Set Theory, An Introduction to Independence Proofs. North-Holland. Amsterdam. 1980. p. xi.
[17] Robert S Wolf, A Tour through Mathematical Logic. The Mathematical Association of America. 2005. p.29.
[18] Haskell B. Curry, Remarks on the definition and nature of mathematics. Reprinted in Benacerraf and Putnam, Eds. Philosophy of Mathematics, Selected Readings. Second edition. Cambridge University Press. Cambridge. 1987. p. 203.
[19] Kenneth Kunen, Set Theory, An Introduction to Independence Proofs. North-Holland. Amsterdam. 1980. p. 7.
[20] John Passmore, A Hundred Years of Philosophy, Penguin. Second Edition, 1966, p. 398.
[21] John Passmore, A Hundred Years of Philosophy, Penguin. Second Edition, 1966, p. 169.
[22] Edmund Husserl, Formal and Transcendental Logic, Translated by Dorion Cairns, 1969. MartinusNijhoff. The Hague. p.16.

[23] Henri Poincaré, Mathematics and Logic (1914), in Science and Method, translated by Andrew Pyle. Routledge, London 1966.

[24]http://www.genealogy.ams.org/index.php

[25] This is not a place to discuss the other related question of intuitionism and classical logic, and the debate over the law of excluded middle. This issue is sometimes conflated with the is mathematics discovered or created question.

[26] Here I am not referring to the program of Hilbert, which I prefer to call "Hibertism", but to the formalism represented by Curry.

[27] Paul Benacerraf, What numbers could not be, reprinted in Benacerraf and Putnam (Eds), Philosophy of Mathematics, Selected Reading, Prentice-Hall, 1964, p. 291.

[28] Ibid, p. 291.

[29]http://plato.stanford.edu/archives/spr2008/entries/nominalism-metaphysics/

[30] Ibid.

[31] Ibid.

[32] Ludwig Wittgenstein, Philosophical Investigations, Trans. G.E.M. Anscombe, Basil Blackwell, Oxford, 1958, p. 32.

[33] Charles Parsons, What is the Iterative Conception of Set? Reprinted in Benacerraf and Putnam, Eds. Philosophy of Mathematics, Selected Readings. Second edition. Cambridge University Press. Cambridge. 1987. p. 507.

[34]Joseph Warren Dauben, Georg Cantor – His Mathematics and Philosophy of the Infinite, Harvard University Press, Chapter 10 – Foundations of Cantorian Set Theory.

[35]Prima facie, this is refuted on technical grounds: every first-order logic has a model which is a Boolean algebra, and every such Boolean algebra has a model that is a set; the point being that first-order logic is just one abstract structure, and once it is given, it is in correspondence with any other number of abstract structures.

[36] Michael Potter, Set Theory and Its Philosophy, Oxford University Press, 2004.

[37]Wang is the author of a biography of Gödel (Hao Wang, A Logical Journey From Godel to Philosophy, The MIT Press, Cambridge, Massachusetts. 1996). Gödel was the last logician to openly declare for Plato. (He believed in a spirit world.) Wang is clearly sympathetic to Gödel. So Wang is a good candidate for a Neo-Platonist.

[38] Hao Wang, The Concept of Set, Reprinted in Benacerraf and Putnam, Eds. Philosophy of Mathematics, Selected Readings. Second edition. Cambridge University Press. Cambridge, p. 531.

[39] Charles Parsons, What is the Iterative Conception of Set? Reprinted in Benacerraf and Putnam, Eds. Philosophy of Mathematics, Selected Readings. Second edition. Cambridge University Press. Cambridge. 1987. p. 503.

[40] "Class" here is a synonym of "set"; an example of an attribute of a physical object is "zoological species".

[41] W.V.O. Quine, On What There Is. Reprinted in *From a Logical Point of View*, Harvard University Press, Cambridge Massachusetts, 1953, p.18. This essay is a good evidence of the war that was waged against the idealists. Originally published in 1948, Quine satirises in it the positions of McX = McTaggart (died 1925) and Wyman = Meinoing (died 1920). The fact that he writes so long after both philosophers have died indicates that this was a mopping-up operation. Yet, the issue of universals remains so much a "problem" that a cogent solution was felt to be needed. Quine's teacher was Whitehead, who crossed over to the side of the Gods.

[42] "For all *p*", and "There exists a *p*" respectively.

[43] W.V.O. Quine, Reification of Universals. Reprinted in *From a Logical Point of View*, Harvard University Press, Cambridge Massachusetts, 1953, p.123.

[44] W.V.O. Quine, Reification of Universals. Reprinted in *From a Logical Point of View*, Harvard University Press, Cambridge Massachusetts, 1953, pp.121- 2.

[45] See citation above.

[46]One of these is the Dedekind pigeon-hole principle, which states that it is not possible to fit $n+1$ items into n boxes without putting two items into one box. I also believe that we see an intentional logic at work based on the exchange of different ways of referring to the same object in abstract algebra. An example is Lagrange's Theorem, about which Beeson writes, "A test problem here is the theorem of LaGrange in group theory, according to which the coset of a subgroup of a finite group all have the same number of elements, i.e., are in one-to-one correspondence. The proof of this theorem is very simple by ordinary mathematical standards, yet it seems to be too difficult for automated deduction; and the bottleneck seems to be that several different

data types are involved." Michael J. Beeson, Computerizing Mathematics: Logic and Computation, 1988 in Rolf Herken (Ed), The Universal Turing Machine. A Half-Century Survey. Oxford University Press, 1988, p. 212.

[47] It is popularly thought that limits and the actual infinite has been expunged from analysis by the theory of ratios, hence an appendix is needed to clarify this point.

[48] David M. Bressoud, A Radical Approach to Lebesgue's Theory of Integration, Cambridge University Press, 2008.

[49] To my knowledge this distinction was first introduced by Bertrand Russell in *The Problems of Philosophy* in his chapter on Idealism. It became the cornerstone of the phenomenalists' rejection of Berkeley's argument in favour of idealism.

[50] Galileo Galilei, The Assayer, 1623: "Philosophy is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth."

[51] Michael Dummet, Frege Philosophy of Language, Duckworth, Second Edition, 1981, Chapter 14, Abstract Objects.

[52] I am not suggesting that this is a trivial question.

[53] C. G. Jung, Memories, Dreams, Reflections, Ed. Aniela Jaffé, Trans. Richard and Clara Winston, Collins, 1961, p.330

[54] Ibid. pp. 356 – 7

[55] The history of C19th analysis would support these conclusions. Cauchy believed in the existence of *infinitesimals* and explicitly constructed his analysis on that basis. Gauss did seek to legislate against the actual infinite and infinitesimals, but Bolzano, Weierstass, Dirichlet and Dedekind all continued to use and develop one or other of these notions, for the simple good reason – that analysis cannot be conducted without such concepts. Cantor technically expunged the infinitesimal from analysis and was scathing about contemporary attempts to found analysis upon it – for example, the work of Du BoisReymond – but he substituted the actual infinite in its place. The conclusion here is that historical *C19th analysis is wedded to one or other or both of the concepts of infinitesimal or actual infinite and is irreducibly what we would call second-order in conception.*